Lecture 3, 06/11/13

The Dual Space

Definitions.

1. A subset $B \subset V$ is called bounded if $\forall U \subset V$ open, $\exists \lambda$ such that $\lambda \cdot U \supset B$ (when the topology on V is given by a sequence of norms n_i, this is equivalent to the demand that B will be bounded with respect to any one of the norms n_i).

2. As a linear space, $V^* = \{f : V \to R : f \text{ is linear and continuous}\}$. There are many topologies we can define on V^*, but we will consider only two topologies. Denoting $U_{\varepsilon,S} = \{f : \forall x \in S,\ f(x) < \varepsilon\}$, we define the bases of the topologies to be:

$$\begin{aligned} \text{For } V_W^* : \ B &= \{U_{\varepsilon,S} : \varepsilon > 0,\ |S| < \infty\} \\ \text{For } V_S^* : \ B &= \{U_{\varepsilon,S} : \varepsilon > 0,\ S \text{ is bounded}\} \end{aligned}$$

   Notice that convergence of sequences in V_W^* is point-wise, and convergence in V_S^* is on bounded sets.

Let V be a Fréchet space, and denote by V_{n_i} the completion of V with respect to the norm n_i, so V_{n_i} is a Banach space. We can "order" the norms n_i by replacing n_i with $\max_{j \leq i} n_j$, and get that V_S^* is a direct limit of these Banach spaces: $V_{n_1}^* \subset V_{n_2}^* \subset ... \subset V_S^* = \underrightarrow{\lim} V_{n_i}$.

Consider the embedding $C_c^\infty(\mathbb{R}) \hookrightarrow C^{-\infty}(\mathbb{R})$, $f \mapsto \xi_f$ (by our definitions, $C^{-\infty}(\mathbb{R}) = (C_c^\infty(\mathbb{R}))^*$).

Exercise 1. *Show that this embedding is dense with respect to the weak topology on $C^{-\infty}(\mathbb{R})$ (as a dual space).*

Exercise* 2. *Show that this embedding is dense with respect to the strong topology on $C^{-\infty}(\mathbb{R})$ (as a dual space).*

Hint for both exercises: Show that δ is in the closure of the image.

Definitions.

1. For $U \subset \mathbb{R}^n$ open, we define $C_c^\infty(U)$ to be the space of smooth functions with compact supports, inside U. "Extending" these function to $\mathbb{R}^n$ by 0, we have a map: $\varphi : C_c^\infty(U) \hookrightarrow C_c^\infty(\mathbb{R}^n)$, keeping in mind that the topology on $C_c^\infty(U)$ is not the induced topology from $C_c^\infty(\mathbb{R}^n)$.

2. $C^{-\infty}(U) \equiv (C_c^\infty(U))^*$.

3. For $U_1 \subset U_2$, we define the map of restriction of distributions $C^{-\infty}(U_2) \to C^{-\infty}(U_2)$ to be the transpose of φ: For $\xi \in C^{-\infty}(U_2)$, $f \in C_c^\infty(U_1)$, $\xi|_{U_1}(f) = \xi(\varphi(f))$. Thus $\xi|_{U_1} \in C^{-\infty}(U_1)$ (since φ is continuous, and composition of continuous is continuous).

Notice that for any $K \subset U$ compact, $C_K^\infty(U) \subset C_c^\infty(U)$ (here the topology on $C_K^\infty(U)$ is indeed the induced topology from $C_c^\infty(U)$). We will prove next that with respect to the restriction of distributions defined above, the distributions form a sheaf.

Lemma. Let $f \in C_c^\infty(U)$, $U = \cup_i U_i$. Then $f = \sum_i f_i$ where $f_i \in C_c^\infty(U_i)$ (notice that $\forall x$, $|\{i : f_i(x) \neq 0\}| < \infty$).

Proof. We can assume that U_i are balls (otherwise, replace each U_i by a union of balls). Denote $K = supp(f)$, it is compact and covered by open balls, so there exists a finite sub-cover: $K \subset \bigcup\limits_{i=1}^{n} U_i = \bigcup\limits_{i=1}^{n} B(x_i, r_i)$. Since the cover is open and K is closed, there exists $\varepsilon > 0$ such that $K \subset \bigcup\limits_{i=1}^{n} B(x_i, r_i - \varepsilon)$. Denote by ρ_i the smooth step function such that $\rho_i|_{B(x_i, r_i-\varepsilon)} = 1$, $\rho_i|_{B(x_i,r_i)^C} = 0$. Since $\forall x \in K$, $\sum_{i=1}^n \rho_i(x) \neq 0$, we can define:

$$f_i = \begin{cases} \frac{\rho_i \cdot f}{\sum_{i=1}^n \rho_i} & x \in K \\ 0 & x \notin K \end{cases}$$

□

Theorem. With respect to the above restriction map, the distributions form a sheaf, i.e., let $U = \cup_i U_i$, then:

1. For every ξ, if $\forall i$, $\xi|_{U_i} = 0$, then $\xi|_U = 0$.

2. Given $\{\xi_i\}$ distributions on $\{U_i\}$ respectively, that agree on intersections (i.e. $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$), there exists a distribution ξ on U, such that $\xi|_{U_i} = \xi_i|_{U_i}$.

Proof. 1. Falls immediately from the lemma.

2. We show the outlines of two proofs:

   (a) Choosing a compact set $K \subset U$, we wish to define $\xi_K : C_k^\infty(U) \to \mathbb{R}$. We fix smooth step functions ρ_i on finitely many U_i that cover K (as done in the lemma) and consider the map $\varphi : \bigoplus\limits_{i=1}^{n} C_{c,K\cap U_i}^\infty(U_i) \to$

$C_K^\infty(U)$. The lemma proves it is onto, and clearly it is continuous. Therefore, by Banach-Schauder theorem, φ is an open map.

$$\begin{array}{ccc} \bigoplus_{i=1}^{n} C_{c,K\cap U_i}^\infty(U_i) & \xrightarrow{\varphi} & C_K^\infty(U) \\ \downarrow & \swarrow & \\ \mathbb{R} & & \end{array}$$

We claim that there exists a third map, and that it is continuous.

(b) Define $\xi(f) = \sum_i \xi_i \left(\frac{\rho_i \cdot f}{\sum_j \rho_j}\right)$ and show that it is linear and continuous, and that $\xi|_{U_i} = \xi_i$.

$\square$

Exercise 3. *Complete the details of at least one of the proofs.*

Definition. *supp*$(\xi) \equiv (\cup \{U : \xi|_U = 0\})$.

Exercise* 4. *Describe* $\overline{C_c^\infty(U)}$.

Solution: $\{f \in C_c^\infty(\mathbb{R}^n) : \forall x \in U,\ \forall$ differential operator $L,\ Lf(x) = 0\}$.

Exercise 5. *Special case: Solve exercise 4, with* $U = \mathbb{R}^n \setminus \mathbb{R}^k$.

Now we wish to describe the space of distributions supported in $\mathbb{R}^k$:

$$V = C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)$$

We start by describing it's dual space:

$$V^* = {C_c^\infty(\mathbb{R}^n)}/{\overline{C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^k)}}$$

We define a descending filtration on V^*:

$$F^i(C_c^\infty(\mathbb{R}^n)) = \left\{f \in C_c^\infty(\mathbb{R}^n) : \forall x \in \mathbb{R}^k,\ \forall \text{ differential } D \text{ s.t.} \deg(D) \le i, Df(x) = 0\right\}$$

This gives us a filtration of the quotient: $F^i\left({C_c^\infty(\mathbb{R}^n)}/{\overline{C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^k)}}\right)$. The corresponding filtration on the dual space V will then be:

$$F_i\left(C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)\right) = \left\{\xi : \langle \xi, f\rangle = 0 \ \ \forall f \in F^i(C_c^\infty(\mathbb{R}^n))\right\}$$

Exercise 6. *Show that this filtration does not cover* $C^{-\infty}(\mathbb{R}^n)$.

Exercise 7. $\forall x\ \exists$ *open neighborhood* $U \ni x$, *such that* $\forall \xi \in C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)\ \exists i, \tilde{\xi} \in F_i\left(C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)\right)$ *such that* $\xi|_U = \tilde{\xi}|_U$ *(i.e. the filtration locally covers).*

Exercise 8. *Consider a smooth function* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *that fixes* $\mathbb{R}^k$. *Show that changing coordinates using* φ *for* $\xi \in F_i$ *we get a distribution in* F_i *(so* F_i *is preserved under change of coordinates:* $\varphi^*(F_i) = F_i$ *, meaning:* $\forall \xi \in F_i,\ \xi(\varphi(f)) \in F_i$*).*

Claim. $\oplus^{F_{i+1}}/_{F_i} \cong \cup_{i=0}^{\infty} F_i$.
Proof: We number the differential operators D_i where i is a multi-index, $i \in \mathbb{N}^{n-k}$ and demand $|i| = \sum_j i_j = \ell$. Consider the space $\oplus_i D_i C^{-\infty}\left(\mathbb{R}^k\right) \equiv V_\ell$ (we derive ξ as a distribution over $\mathbb{R}^n$ and then restrict it to $\mathbb{R}^k$). Then $V_\ell \subset F_\ell$, $V_\ell \cap F_{\ell-1} = 0$ and $F_\ell = V_\ell \oplus F_{\ell-1}$.

Exercise 9. *Show that this decomposition is not invariant under change of coordinates.*

Tensor Product

Exercise 10. *Show that* $Bil(\mathbb{R}, \mathbb{R}) \neq \{\textit{linear maps } \mathbb{R}^2 \to \mathbb{R}\}$.

Equivalent definitions.

1. The tensor product of V, W is defined by a bilinear form:

$$\begin{array}{ccc} V \times W & \xrightarrow{Bil} & L \\ \downarrow & \nearrow & \\ V \otimes W & & \end{array}$$

   This means: $(V \otimes W)^* = Bil\,(V, W)$. In the finite dimensional case, $V \otimes W = Bil\,(V, W)^*$.

2. For formal combinations $V = sp(e_i)$, $W = sp(f_j)$ we define $V \otimes W = sp(e_i \otimes f_j)$. Change of bases here is messy.

3. Definition by a quotient:

$$V \otimes W = ^{sp(v \oplus w:\ v \in V,\ w \in W)} /_{\{(v+v_1)\oplus w - v \oplus w - v_1 \oplus w\} \ldots}$$

Properties. $(V \otimes W)^* = V^* \otimes W^*$, $Hom(V, W) = V^* \otimes W$.