Solution for second week problems

Or Dagmi

1 Find a topological linear space which is not locally convex.

Lets look at $\ell^p = \left\{ \{x_i\}_{i\in\mathbb{N}} \mid \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$ where $0 < p < 1$.
The metric will be defined as follows:

$$d(x,y) = |x-y|_p$$

Where:

$$|x|_p = \sum_{i=1}^{\infty} |x|^p$$

Remarks 1.1 It is not a norm, because it is not homogeneous of degree 1.

Lemma 1.2

$d(x,y) = |x-y|_p$ maintains the triangle inequality.

Proof: Define

$$f(t) := (1+t)^p - 1 - t^p$$

Note that $f(0) = 0$. And:

$$f'(t) = p(1+t)^{p-1} + pt^{p-1} = p\left((1+t)^{p-1} + t^{p-1}\right)$$

Note that $\forall t \geq 0$ we get that $f'(t) < 0$ because: $(1+t)^{p-1} < t^{p-1}$. So $\forall t \geq 0$, f is a decreasing function.
Now, for every $a, b > 0$, denote $t = \frac{a}{b}$. Because the function is decresing then $f(t) \leq 0$. We get:

$$0 \leq f\left(\frac{a}{b}\right) = \left(1 + \frac{a}{b}\right)^p - 1 - \left(\frac{a}{b}\right)^p \Rightarrow 1 + \left(\frac{a}{b}\right)^p \leq \left(1 + \frac{a}{b}\right)^p \Rightarrow b^p + a^p \leq (b+a)^p$$

As required. ■

Claim 1.3

Given the above notations, ℓ^p is not locally convex.

Proof: Assuming that ℓ^p was locally convex, then the convex hull of the δ-ball at 0 must be contained in some r-ball.
Meaning, local convexity requires that for any given δ, there is a r such that:

$$\left| \frac{1}{n}(\delta, 0, \ldots) + \ldots + \frac{1}{n}(\underbrace{0, \ldots, 0, \delta}_{n}, 0, \ldots) \right|_p = \left(\frac{\delta}{n}\right)^p + \ldots + \left(\frac{\delta}{n}\right)^p < r$$

For $n = 1, 2, 3, \ldots$
Note that this is a finite sum for every n.
That is, for any given δ, there is a r such that:

$$n^{1-p} < \frac{r}{\delta^p}$$

But this is impossible because $0 < p < 1$. Thus a contradiction. ■

2 Let V be a locally convex linear topological space, show V is Hausdorff iff $\{0\}$ is closed.

Claim 2.1

If V is Hausdorff then $\{0\}$ is closed.

Proof: Define V_x to be a neighborhood of x that seperates x from 0. That is $x \in V_x$, V_x is open and there exists U_x which is an open set s.t. $0 \in U_x$ and: $U_x \cap V_x = \varnothing$.

Now define:

$$W := \bigcup_{x \in X \setminus \{0\}} V_x$$

It's clear that $W = V \setminus \{0\}$ and that W is open (all V_x are open, so W is open as a unit of open sets). Thus $\{0\}$ is closed. ■

Claim 2.2

If V is a locally convex linear topological space s.t. $\{0\}$ is closed, Then V is Hausdorff.

Proof: Because V is linear, It's enough to show that we can separate $\{0\}$ from every $x \in V \setminus \{0\}$.

Let $W = V \setminus \{x\}$ which is a neighborhood of 0. Since this is a linear topological space, the adition is continous. Meaning there are V_1, V_2 neighborhoods of 0 such that $V_1 + V_2 \subset W$.

Then $x - V_2$ is a neighborhood of x and $V_1 \cap (a - V_2) = \varnothing$. ■

Remarks 2.3 $V_1 \cap (a - V_2) = \varnothing$ because if not, $a \in V_1 + V_2$.

3 Find such a space (locally convex and hausdorff) that is isomoprhic to $\mathbb{F}^n$.

V has finite dimensions, meaning ther is a basis $v_1, \ldots, v_n$ for V. Define:

$$\begin{aligned} f : \mathbb{F}^n &\to V \\ (a_1, \ldots, a_n) &\mapsto \sum_{i=1}^{n} a_i v_i \end{aligned}$$

Note that f is continous, and of course inversible.

We need to show that f is homeomorphism. Meaning it only remains to show that f^{-1} is continous.

It's enough to show that it is continous in 0 (because it is a linear operator).

Define $S := \left\{x \in \mathbb{F}^n \mid \sum_{i=1}^{n} \left|x_i^2\right| = 1\right\}$. This is a compact set. Thus $K = f(S)$ is compact. But V is Hausdorff space, and so it's also close.

It is clear that $0 \notin K$ (because f is one-to-one and $0 \notin S$). Now because V is Hausdorff, There is an open set U s.t. $0 \in U$ and $U \cap K = \varnothing$.

The set $E = f^{-1}(U)$ is therefore disjoint from S. Because f is linear then E is balanced and therefore connected. Thus $E \subset B_0(1)$, meaning that E is bounded.

Hence, there U is a neighborhood of 0 s.t. $f^{-1}(U)$ is bounded. And because it is a linear operator, that means that f^{-1} is continous as required.

Meaning tht f is homeomorphism.

4 Find a seq. complete space which is not complete.

Define: $X := \mathbb{R}^I$ when $|I| > \aleph_0$.

Define $H \subseteq X$ s.t. $H := \left\{\{x_i\}_{i\in I} \mid |\{x_i \neq 0\}| \leq \aleph_0\right\}$. H is dense but it's seq. complete because if we take: $\forall \{x_i\}_{i\in I} \subseteq H$ note that $\exists U$ neigh. of 0 s.t. $\exists n_0 \in \mathbb{N}$. But note that the support of all the sequences together is countable (becuase the support of each one is countable, and there are countable amount of sequences). So the limit is also in H.

5

From the universal quality we can deduce the previous defenition simply by choosing $W = \overline{V}$ and $\phi_W = \text{Id}$.

On the other way, We are assuming that $\overline{V}$ is the completion of V, hence exists an embedding $i : V \to \overline{V}$ s.t. $i(V)$ is dense in $\overline{V}$ and isomoprhic to V.

Let W be a complete space and $\psi : V \to W$. Note that $i(V)$ is isomorphic to V, hence it is invertible on the image. So we can define: $\phi_W = \psi \circ i^{-1}$ and we get:

$$\psi \equiv \phi_W \circ i$$

as required.

6

7 Let V be a locally convex space, and let $f : W \to \mathbb{F}$ be a continuous function from $W \subseteq V$. Show that f can be lifted.

Consider the projection: $p_W : V \to W$. Take a look at $\varphi = f \circ p_W$.

But p_W is continous, thus φ is continuous.

Note that for every $w \in W$ we get $p_W(w) = w$, hence: $\varphi(w) = f(w)$.

8 Fréchet...

1. S_1.
2. Sequence of seminorms.
3. Met.

$3 \Rightarrow 1$:

Balls with rational radiuses and centers of rational points is countable.

$1 \Rightarrow 2$:

For every U in the basis, we take $C = \frac{1}{2}(U) - \frac{1}{2}(U)$ which is convex and a subset of U. Take $n_n(x) = \inf\left\{\lambda \mid \frac{x}{\lambda} \in U\right\}$, Note that n_λ is finite because $C \subseteq U$, $0 \in C$ and C is convex. Meaning there is a small ball centered around 0 which is a subset of C. So for large enough λ we get $\frac{x}{n} \in C \Rightarrow \frac{x}{n} \in U$.

$2 \Rightarrow 3$:

$$d(x, y) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|x - y\|_k}{1 - \|x - y\|_k}$$

9 Let V be a Fréchet space, V is complete $\iff$ V is sequential complete.

Let $\varphi : V \to W$, because V is complete then $\varphi(V) = \overline{\varphi(V)}$.

Let $\{x_i\}$ be a cauchy sequence. Because $\varphi(V)$ is an isomorphism, $\{\varphi(x_i)\}$ is also a cauchy sequence. But $\varphi(V)$ is closed, therefore exists $c \in \text{Im}\varphi$ s.t. $\varphi(x_i) \to c$.

But because $c \in \text{Im}\varphi$, exists $x \in V$ s.t. $\varphi(x) = c$. Therefore x_i converges to x (otherwise $\varphi(x_i)$ couldn't converge to $\varphi(x)$).

10 $C^{\infty}\left(S^{1}\right) \cong$Sequences that goes to 0 faster than any polynomial

Simply by fourier sequence. Every function in $C^{\infty}\left(S^{1}\right)$ can be represented uniquley by a series of coeefficents to cos and sins that goes faster to 0 than any polynomial, And of course every sequence of that kind converge into a smooth function.