Solution for problems

Or Dagmi

1 Prove ostrowski's theorem.

A proof can be found here: http://en.wikipedia.org/wiki/Ostrowski%27s_theorem.

2 $|\,|_\alpha \sim |\,|_\beta \iff \exists c\ |\,|_\alpha^c = |\,|_\beta$.

Assume that $|\,|_\alpha \sim |\,|_\beta$, And let $|x|_\alpha < 1$. Note that a sequence $x_n \overset{|\,|_\alpha}{\to} x \iff x_n \overset{|\,|_\beta}{\to} x$.
And: $|x^n|_\alpha \to 0$ Thus: $|x^n|_\beta \to 0$. Meaning: $|x|_\beta < 1$ as well.
So we got: $|x|_\alpha < 1 \iff |x|_\beta < 1$.
Let a be s.t. $|a|_\alpha \neq 0, 1$. If $|a|_\alpha < 1$ then also $|a|_\beta < 1$ and if $|a|_\alpha > 1$ then so is $|a|_\beta$ (because we can look at $\left|a^{-1}\right|_\alpha < 1$).
So there exists c s.t. $|a|_\alpha = |a|_\beta^c$. Now for arbitrary x s.t. $|x|_\alpha \neq 0, 1$ there exists $t \in \mathbb{R}$ s.t.:

$$|x|_\alpha = |\alpha|_\alpha^t$$

Suppose ${}^m/_n \in \mathbb{Q}$ s.t. ${}^m/_n < t$ then:

$$|a|_\alpha^{m/n} < |x|_\alpha \Rightarrow |a|_\alpha^m < |x^n|_\alpha \Rightarrow \left|\frac{a^m}{x^n}\right|_\alpha < 1$$

So by what we've shown:

$$\left|\frac{a^m}{x^n}\right|_\beta < 1$$

So we've got:

$$|a|_\beta^{m/n} < |x|_\beta$$

Similarly, with ${}^{m'}/_{n'} > t$ we can show that:$|a|_\beta^{m'/n'} > |x|_\beta$.
Because the absoulute value is continous we get that:

$$|a|_\beta^{m/n} < |x|_\beta < |a|_\beta^{m'/n'}$$

So: $|x|_\beta = |a|_\beta^t$. Thus:

$$|x|_\alpha = |a|_\alpha^t = |a|_\beta^{ct} = |x|_\beta^c$$

As required.
Let $i \in \{\alpha, \beta\}$ and $B_{r,i}(x) = \{y \mid \ |y - x|_i < r\}$ be the open r-ball.

Note that:

$$\begin{aligned}B_{r,\beta}(x) &= \left\{y \mid \ |y-x|_{\beta} < r\right\} \\ &= \left\{y \mid \ |y-x|_{\alpha}^{1/c} < r\right\} \\ &= \left\{y \mid \ |y-x|_{\alpha} < r^{c}\right\} \\ &= B_{r^{c},\alpha}(x)\end{aligned}$$

So, let U be an open set w.r.t $|\,|_{\alpha}$, and let $x \in U$. Then for some $r > 0$, $B_{r,\alpha}(x) \subseteq U$. So, from the above computation: $B_{r^{1/c},\beta} \subseteq U$.
Thus U is open w.r.t $|\,|_{\beta}$.

3 $\mathbb{Q}_p = \{\dots x_{-n} \dots x_0.x_1 \dots x_k \mid \ x_i \in \{0, \dots, p_{-1}\}\}$.

It's clear that $A = \{\dots x_{-n} \dots x_0.x_1 \dots x_k \mid \ x_i \in \{0, \dots, p_{-1}\}\} \hookrightarrow \mathbb{Q}_p$, by:

$$\varphi(x) = \sum x_n p^{-n}$$

This is a infinite sum, but only from one side. It is converge because $x_n p^{-n} \to 0$ (see next question).
Now we will build $\mathbb{Q}_p \hookrightarrow A$. We will do that by taking $\mod p$ on the number and divide it by p every time.

4 $\exists x \ \sum x_n \to x \iff |x_n|_p \to 0$.

It's clear that if $\sum x_n$ converges then $|x_n|_p \to 0$.
We want to the other direction.
Suppose $x_n \to 0$, Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for $n > N$, $|x_n|_p < \varepsilon$.
Denote:

$$S_n = \sum_{i=1}^{n} x_n$$

Thus, for $n, m > N$ we get:

$$|S_n - S_m|_p = \left| \sum_{i=\min(m,n)+1}^{i=\max(m,n)} x_n \right|_p \leq \max_{i>N} |x_n|_p < \varepsilon$$

so the partial sums are Cauchy and consequently converge.

5 $\forall y \in B_r(x) \quad B_r(x) = B_r(y)$.

Two
Let $y \in B_r(x)$. Note that:

$$B_r(x) = \left\{z \mid \ |z-x|_p < r\right\}$$

Assume exists $z \in B_r(x)$ s.t. $z \neq B_r(y)$, Then: $|z-y|_p > r$ and $|z-x|_p < r$.
But:

$$|z-x|_p = |(z-y)+(y-x)|_p = \max\left(|y-x|_p, |z-y|_p\right) > r$$

A contradiction. (The other direction is similar).

6 $\mathbb{Z}_p$ is homeomorphic to the cantor set.

We will show it for $\mathbb{Z}_2$:

The elements of each of these sets can be denoted by sequences (x_n) with each $x_i \in \{0,1\}$, In particular, each 2-adic integer can be represented in canonical form as $\sum\limits_{n=0}^{\infty} x_n 2^n$.

If two sequences in this representation, (x_n) and (y_n), differ in the jth term and no other term before j, the distance between them is:

$$|(x_n)-(y_n)|_2 = \left|\sum_{k=j}^{\infty}(x_k-y_k)\,2^k\right| = 2^{-j}$$

If two elements of the Cantor set under this representation differ in the jth term but no earlier term, then the distance between them is bounded:

$$|(x_n)-(y_n)|_\infty = \left|\sum_{k=j}^{\infty}\frac{2\,(x_k-y_k)}{10^k}\right|_\infty \le \sum_{k=j}^{\infty}\left|\frac{2\,(x_k-y_k)}{10^k}\right|_\infty \le \sum_{k=j}^{\infty}\left|\frac{2}{10^k}\right|_\infty = \frac{2}{9}\cdot 10^{-j} < \infty$$

We can define a bijection $\varphi : \mathcal{C} \to \mathbb{Z}_2$ such that φ sends an element determined by the sequence (x_n) in the Cantor set to the element determined by the same sequence in $\mathbb{Z}_2$. ince each element of each set is uniquely expressible in this way, this mapping is clearly a bijection. We prove that it is also continuous using the metric definition of continuity. Fix some $x \in \mathcal{C}$. Let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$2^{-(N+1)} < \varepsilon \le 2^{-N}$$

We can choose $\delta = \frac{2}{9}\cdot 10^{-N}$ s.t.:

$$|x-y|_\infty < \delta \Rightarrow |\varphi(x)-\varphi(y)|_2 < \varepsilon$$

For any $y \in \mathcal{C}$. In particular, this δ implies that x and y have at least the first $N+1$ terms of their representative equences in common, forcing $|x-y|_2 < 2^{-N+1} < \varepsilon$. Thus φ is continous.

Remains to show that φ^{-1} is continous. Fix $x \in \mathbb{Z}_2$, Let $\varepsilon > 0$ we can find N s.t.

$$\frac{2}{9}10^{-N} < \varepsilon \le \frac{2}{9}10^{-N+1}$$

Choose $\delta = 2^{-(N+1)}$ and we get:

$$|x-y|_2 < \delta \Rightarrow \left|\varphi^{-1}(x)-\varphi^{-1}(y)\right|_\infty < \infty$$

So φ^{-1} is also continous.

Thus $\mathbb{Z}_2$ and $\mathcal{C}$ are homeomorphic as required.

Now we will prove the following lemma:

Lemma 6.1

$\mathbb{Q}_p$ is totally disonnected.

Proof: Let $a \in \mathbb{Q}_p$. The connected component C_a of a is eqaul to $\{a\}$.

Let a be arbitrary and suppose $C_a \supsetneqq \{a\}$, therefore there exists $n \in \mathbb{N}$ s.t. $B_{(a,p^{-n})} \cap C_a \neq C_a$, But then:

$$C_a = \left(B_{(a,p^{-n})} \cap C_a\right) \cup \left(\left(\mathbb{Q}_p \backslash B_{(a,p^{-n})}\right) \cap C_a\right)$$

which is the disjoint union of two open subsets. Therefore C_a os mpt cpmmected. which is a contradiction. ■

Theorem 6.2

Any compact perfect totally disconnected subset E of the real line is homeomorphic to the Cantor set.

Proof: Denote $m = \inf E$ and $M = \sup E$.

We will build $F : [m.M] \to [0,1]$ s.t. F maps E homeomorphic on $\mathcal{C}$.

We will build F on the complements $[m,M] \setminus E \to [0,1] \setminus C$, and by continuity to a map $F : [m,M] \to [0,1]$.

$[m,M] \setminus E$ is the disjoint union of countably many open intervals, and the same is true for $[0,1] \setminus C$. Let $\mathcal{I}$ be the collection of the intervals whose union is $[m,M]$ and let $\mathcal{J}$ be the collection whose union is $[0,1] \setminus \mathcal{C}$. We shall build a bijection: $\Theta : \mathcal{I} \to \mathcal{J}$.

Let $I_1 \in \mathcal{I}$ be an interval of maximal length and define:

$$\Theta(I_1) = ({}^{1}\!/\!{}_{3}, {}^{2}\!/\!{}_{3})$$

Next, choose intervals $I_{2,1}$ and $I_{2,2}$ to the left and right of I_1 s.t they have maximal length and define:

$$\begin{aligned} \Theta(I_{2,1}) &= ({}^{1}\!/\!{}_{9}, {}^{2}\!/\!{}_{9}) \\ \Theta(I_{1,2}) &= ({}^{7}\!/\!{}_{9}, {}^{8}\!/\!{}_{9}) \end{aligned}$$

Continuing this process defines Θ on the whole set $\mathcal{I}$, Since $\mathcal{I}$ contains only finitely many sets of length greates then some fixed $\varepsilon > 0$, and since two intervals in $\mathcal{I}$ or in $\mathcal{J}$ have different endpoints (as E and C are perfect). It is clear from the construction that Θ is bijective and order preserving.

Define F as follows:

For $I \in \mathcal{I}$: $F|_I : I \to \Theta(I)$ is a unique linear increasing map. Because E and C are totally disconnected, they are nowhere dense. Thus there exists at most one continuation $F : [m,M] \to [0,1]$. Now, From our construction of Θ:

$$F(x) = \sup \{F(y) : \; y \notin E, \; y \leq x\}$$

Let $f : F|_E$. Note that $f : E \to C$ is a monotone increasing, continous bijection and we need to show that $g := f^{-1}$ i continous.

Note that g is again monotone increasing. Let $x \in C$ and $x_n \to x$. This sequence contains a monotone subsequence and thus we may assume, wlog, that the sequence x_n is monotone increasing.

Clearly

$$y = \lim_{n\to\infty} g(x_n) = \sup_{n\geq 1} g(x_n) \leq g(x)$$

Assume $y < g(x)$, since E is closed, we have $y \in E$ and $g^{-1}(y) < x$. This implies that $y < x_n$ for large n and by monotonicity $y < g(x_n)$. A contradiction to the definition of y, Thus g continous. ■

The same type of proof will work for all $\mathbb{Z}_p$ showing it is homeomorphic to $\mathcal{C}^{(p)}$ defined by:

$$\mathcal{C}_n^{(p)} := \mathcal{C}_{n-1}^{(p)} \cap \left((2p-1)^{-n} \left(\bigcup_{k\in\mathbb{Z}} [2k, 2k+1] \right) \right)$$

And define:

$$\mathcal{C}^{(p)} := \bigcap_{n\geq 0} \mathcal{C}_n^{(p)}$$

Note that $\mathcal{C}^{(p)}$ is perfect for every p, thus $\mathbb{Z}_p$ is perfect for every p, and by the theorem, because it is also totally disconnected, $\mathbb{Z}_p$ is homeomorphic to the Cantor set as required.

7 $\mathbb{Z}_p \cong \varprojlim \mathbb{Z}/p^n\mathbb{Z}$

Note that $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$

So we got $\mathbb{Z}_p \mapsto \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, We will build another map from $\varprojlim \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p$.

8 $\mathbb{Q}_p \cong \mathcal{C} \setminus \{1\}$.

Note that $\mathbb{Q}_p$ is a countable union of Cantor sets by taking $\mathbb{Q}_p/\mathbb{Z}_p$, and so it is trivial that it is homeomorphic to $\mathcal{C} \setminus \{1\}$.

9 Haar theorem for $\mathbb{Q}_p$

Define μ s.t. : $\mu(B_1(0)) = 1$, Note that for every ball $B_p(0)$ we can cover it with distinctivly p unit balls.

Because μ need to be σ-additive and invariant to translations, we get that $\mu(B_p(0)) = p\mu(B_1(0)) = p$.

Now note that the definition of $\mu(B_1(0))$ defines μ, hence every other haar measure is only a differ by a constant multiplication.

10 $\alpha(a) = |a|_p$

Define $\alpha(a) = \frac{\mu_a}{\mu}$, We want to show that this is $|a|_p$. Because we know that μ_a is haar measure, then we want to know what is the value on the unit ball,Note that $|ax| = |a||x|$:

$$\frac{\mu(a \cdot B(0,1))}{\mu(B(0,1))} = \frac{\mu(B(0,a))}{\mu(B(0,1))} = |a|$$

Because there are p^n unit balls inside a ball with radius p^n.

11 find a l-space X which is countable at ∞ and an open subset $U \subset X$ that is not countable at ∞.

Take a look at $\{0,1\}^{\mathbb{R}}$, It is compact because of Tichonof thm. But if we exclude 0, it is not countable at ∞.

12 Every l-space metrizable and countable at ∞ space X, is isomorphic to 1 of the 3 spaces, cantor set, cantor set-$\{1\}$ and discrete set.

Follows from the theorem at question 6, as l-spaces are totally disconnected