Generalized Functions: Homework 6

Exercise 1. *Let X ba an ℓ-space, so $C^\infty(X)$ separates points.*

Solution: Let $x, y \in X$. Since X is Hausdorff, there exist $U, V \subset X$ open, such that $x \in U,\ y \in V$ and $U \cap V = \emptyset$. Since X has a basis of open compact sets, we can assume that U, V are open and compact (and thus closed, since X is Hausdorff). Define $f : X \to X$ by:

$$f(z) = \begin{cases} x & z \in U \\ y & otherwise \end{cases}$$

So for every $z \in X$, if $x \in U$ take any $U_z \ni z$ open, then $\tilde{U} = U_z \cap U$ is also open, contains z and $f|_{\tilde{U}} = x$ constant. If $z \notin U$ then take $U_z \ni z$ open, so $\tilde{U} = U_z \cap (X \setminus U)$ is open (since U is closed) and $f|_{\tilde{U}} = y$ constant. This means that $f \in C^\infty(X)$ separates x and y as required.

Exercise 2. *Show that $S^*(X)$ is a sheaf with respect to the restriction map.*

Solution: Let $U = \cup_i U_i$. Denote by $\varphi : S(U_i) \to S(U)$ the extension by zero. Then $\varphi_* : S^*(U) \to S^*(U_i)$, $(\varphi_* \xi)(f) = \xi(\varphi \circ f)$ is the restriction map of distributions.
Lemma: Given a compact set $K \subset X$ and an open (finite) cover $K \subset \cup_{i=1}^n U_i$, there exists a partition on unity on K, sub-ordinate to the open cover.

Proof. We choose a refinement cover of open compact sets: For every $x \in K \cap U_i$ we take open compact $x \in K_{i,x} \subset U_i$, so

$$K \subset \bigcup_{i=1}^n \left(\bigcup_{x \in K \cap U_i} K_{i,x} \right)$$

Since K is compact, there exists a finite sub-cover, $K \subset \cup_{i,j} K_{i,j}$, which is a refinement of the original cover. We can assume that this union is disjoint (taking $\tilde{K}_{i,j} = K_{i,j} \setminus (\cup_{l<i,k<j} K_{l,k})$). Define:

$$\rho_i(x) = \begin{cases} 1 & x \in K_{i,j} \text{ for some } j \\ 0 & \text{otherwise} \end{cases}$$

These are smooth functions (locally constant), *supp*$(\rho_i) \subset U_i$ and $\sum \rho_i = 1$. ∎

1. Let $\xi \in S(U)$, and suppose $\xi_i = \xi|_{U_i} = 0$. For every $f \in S(U)$, let $\{\rho_i\}_i$ be a partition of unity of *supp*(f), sub-ordinate to the cover $\{U_i\}$. So $f = \sum_i \rho_i f$, where each $\rho_i f$ "comes from" a distribution in $S(U_i)$. Thus:

$$\xi(f) = \sum_i \xi(\rho_i f) = \sum_i \xi_i \left(\varphi^{-1}(\rho_i f) \right) = 0$$

2. Let $\xi_i \in U_i$ such that $\forall i, j\ \xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$. Define:

$$\xi(f) = \sum_i \xi_i \left(\frac{\rho_i \cdot f}{\sum_j \rho_j} \right)$$

Clearly ξ is linear, and for $h \in S(U_k)$:

$$\xi|_{U_k}(h) = \xi(\varphi \circ h) = \sum_i \xi_i \left(\frac{\rho_i \cdot \varphi \circ h}{\sum_j \rho_j} \right) \underset{\rho_i \varphi h = \delta_{ik} h}{=} \xi_k(h)$$

i.e. $\xi|_{U_k} = \xi_k$.

Exercise 3. *Show that the map $\psi : S(X) \to S(Z)$ is onto.*

Solution: For $f \in S(Z)$, denote by $Y \subset Z$ the compact (and open) support of f. Since the topology on Z is the induced topology from X, there exists open U (in X) such that $Y = U \cap X$. For every $x \in Y$ we take open compact $x \in U_x \subset U$, and get a cover $Y \subset \cup_x U_x$. There exists a finite sub-cover, as Y is compact: $Y \subset \bigcup\limits_{i=1}^{n} U_i \equiv \tilde{Y}$. The set $\tilde{Y}$ is open and compact. We define $\tilde{f} \in S(X)$ in the following way:

$$\tilde{f}(x) = \begin{cases} f(y) & x \in U_y \subset \tilde{Y} \\ 0 & x \in X \setminus \tilde{Y} \end{cases}$$

Exercise 4. *Let $L \subset V$ be vector spaces. Show that $\forall f : L \to K,\ \exists g : V \to K$ such that $g|_L = f$.*

Solution: Consider the collection of pairs (L_α, f_α) such that $L \subseteq L_\alpha \subseteq V$ and $f_\alpha|_L = f$, with the partial order:

$$(L_\alpha, f_\alpha) < (L_\beta, f_\beta) \iff L_\alpha \subset L_\beta,\ f_\beta|_{L_\alpha} = f_\alpha$$

Given a chain $\{(L_\alpha, f_\alpha)\}_{\alpha \in J}$ there exists an upper bound: (L_∞, f_∞) where $L_\infty = \cup_\alpha L_\alpha$ and $f_\infty(x) = \{f_\alpha(x) : x \in L_\alpha\}$. Therefore, by Zorn's lemma there exists a maximal element $(L_{\alpha_0}, f_{\alpha_0})$. Suppose $V \neq L_{\alpha_0}$, the there exists $y \in V \setminus L_{\alpha_0}$. Consider $F = \mathrm{span}\{L_{\alpha_0}, y\}$ and search for $h \in F^*$ such that $h|_{L_{\alpha_0}} = f_{\alpha_0}$. But, by linearity, for $x \in L_{\alpha_0}$,

$$h(x + \lambda y) = f_{\alpha_0}(x) + \lambda \tilde{h}(y)$$

Choosing for example $\tilde{h}(y) = 1$ we get that $h \in F^*$ and:

$$(L_{\alpha_0}, f_{\alpha_0}) < (F, h)$$

in contradiction to the maximality of $(L_{\alpha_0}, f_{\alpha_0})$. Thus we get $L_{\alpha_0} = V,\ g \equiv f_{\alpha_0}$.

Exercise 5. *Show that the map* $\varphi : S(X) \otimes S(Y) \to S(X \times Y)$ *defined by:*

$$f_1 \otimes f_2 \mapsto f(x,y) = f_1(x) \cdot f_2(y)$$

is well defined and is a linear isomorphism.

Solution: By linearity of φ, it is enough to check for the basic elements $f_1 \otimes f_2$.

- Well defined: Need to show that $\varphi(f_1 \otimes f_2) \in S(X \times Y)$. Indeed,$f$ is compactly supported, since $supp(f) = supp(f_1) \times supp(f_2)$ is compact. f is locally constant:

$$\begin{gathered} \forall (x,y) \in X \times Y \\ \exists U_1 \subset X,\ x \in U_1,\ \text{s.t. } f_1|_{U_1} = c_1 \\ \exists U_2 \subset Y,\ y \in U_2,\ \text{s.t. } f_2|_{U_2} = c_2 \end{gathered}$$

  Denoting $U = U_1 \times U_2 \subset X \times Y$ open, we get:

$$(x,y) \in U,\ f|_U = f_1|_{U_!} \cdot f_2|_{U_2} = c_1 \cdot c_2$$

  i.e. f is indeed locally constant.

- φ is injective: Take $0 \neq \sum_i f_i \otimes g_i \in S(X) \otimes S(Y)$, then we can assume that f_i are linearly independent and that g_i are non-zero.

$$\varphi\left(\sum_i f_i \otimes g_i\right)(x,y) = \sum_i f_i(x) \cdot g_i(y)$$

  Taking y such that $g_1(y) \neq 0$, we get (for linearly independence of $\{f_i\}$) that there exists x such that the function above does not vanish, i.e.,

$$\varphi\left(\sum_i f_i \otimes g_i\right) \neq 0$$

- φ is surjective: Consider $g(x,y) \in S(X \times Y)$ with compact support K. Since g is locally constant, $\forall p \in X \times Y$, $\exists \tilde{U_p}$ open such that $g|_{\tilde{U_p}}$ is constant. Since we have a basis of open compact product sets, we can assume $\tilde{U_p} = U_x \times V_y$ open and compact. We got an open cover $K \subset \cup_{(x,y) \in K} U_x \times V_y$, therefore there exists a finite open compact product sub-cover:

$$K \subset \cup_{i=1}^n U_{x_i} \times V_{y_i}$$

  such that g is constant on every set in this cover. Subtracting from every set in the cover the previous sets we get a disjoint open compact cover. Since every set which is a difference between two product set is a finite disjoint union of product sets, by splitting these sets we have an open compact product disjoint cover, such that g is constant on every set. Let

$f_1^i(x) \equiv 1$ in U_{x_i}, and zero outside (i.e. $supp(f_1^i) = U_{x_i}$), and $f_2^i(y) = g(U_{x_i} \times V_{y_i})$ in V_{y_i} and zero outside (i.e. $supp(f_2^i) = V_{y_i}$). Then:

$$g(x,y) = \sum_{i=1}^{n} f_1^i(x) \cdot f_2^i(y) = \varphi\left(\sum_{i=1}^{n} f_1^i \otimes f_2^i\right)$$

for $\sum_{i=1}^{n} f_1^i \otimes f_2^i \in S(X) \otimes S(Y)$, meaning φ is onto.

Exercise 6. *Consider the map $\psi : S^*(X) \otimes S^*(Y) \to S^*(X \times Y)$, such that for every $f \otimes g$, $\psi(f \otimes g)$ is a functional on $S(X) \otimes S(Y) = S(X \times Y)$:*

$$\forall h_1 \otimes h_2 \in S(X) \otimes S(Y),\ \psi(f_1 \otimes f_2)(h_1 \otimes h_2) = f_1(h_1) \cdot f_2(h_2)$$

Show that ψ is injective with dense image.

Solution: By linearity of ψ, it is enough to check for the basic elements $f_1 \otimes f_2$.

- ψ is injective: Take $\sum f_i \otimes g_i \in S^*(X) \otimes S^*(Y)$ such that f_i are linearly independent and $g_i \neq 0$, then the joint kernel of $f_2, ..., f_n$ is not empty (otherwise we would get an embedding from $S^*(X) \otimes S^*(Y)$ to $\mathbb{C}^{n-1}$ in contradiction, since the dimension of $S^*(X) \otimes S^*(Y)$ is higher). Denote by $K = \bigcap_{i=2}^{n} \ker f_i$ and consider the quotient ${}^{S(X)}/_{K}$. Then $f_2, ..., f_n$ form a basis for this quotient space. If f_1 vanishes on K, we would get that f_1 is a linear combination of $f_2, ..., f_n$, in contradiction to our assumption. Therefore, there exists $v \in K \subset S(X)$ such that $f_1(v) \neq 0$. Taking $w \in S(Y)$ such that $g_1(w) = 1$ we get:

$$\left(\sum f_i \otimes g_i\right)(v \otimes w) \neq 0$$

- ψ has a dense image: Using the previous exercise, we actually need to show that the map $\psi : V^* \otimes W^* \to (V \otimes W)^*$ has a dense image. Notice that a subspace of a dual space is dense if and only if the only element this subspaces vanishes on is the zero element. Take any $\sum v_i \otimes w_i \in V \otimes W$ (like before we assume v_i are linearly independent and w_i are non-zero), then $F = \psi(f \otimes g)$ acts on this element by:

$$F\left(\sum v_i \otimes w_i\right) = \sum f(v_i) \cdot g(w_i)$$

  Taking f such that $f(v_j) = \delta_{1j}$ (possible since v_i are linearly independent) and g such that $g(w_1) = 1$, we get $F\left(\sum v_i \otimes w_i\right) \neq 0$, meaning $Im(\psi)$ is dense in $(V \otimes W)^*$.

Exercise 7. *Show the following linear isomorphisms:*

1. $S(X, V) \cong S(X) \otimes V$.

2. If V is finite-dimensional: $S^*(X,V) \cong S^*(X) \otimes V^*$.

Solution:

1. Consider the map $\varphi : S(X) \otimes V \to S(X,V)$, $\varphi(f \otimes v) = f \cdot v$. It is well-defined, linear and injective (as before, we assume that v_i are linearly independent and that f is non-zero, then $\sum f \cdot v_i \neq 0$). In order to define a map in the other direction, we choose a basis of V, $\{e_i\}$ and define $\psi : S(X,V) \to S(X) \otimes V$ by:

$$\psi(F) = \sum_i \langle F(x), e_i \rangle \otimes e_i$$

This is a linear map (and continuous). It is easy to see that $\psi \circ \varphi = id$ and $\varphi \circ \psi = id$. This yields the required isomorphism.

2. Consider the map $\Phi : S^*(X) \otimes V^* \to S^*(X,V)$ that sends every $f \otimes \ell \in S^*(X) \otimes V^*$ to $F \in S^*(X,V) \cong (S(X) \otimes V)^*$ where:

$$F(h \otimes v) = f(h) \cdot \ell(v)$$

Clearly Φ is linear and injective (the same proof as in exercise 6). Like before, we present the map in the other direction: $\Psi : S^*(X,V) \to S^*(X) \otimes V^*$, $\Psi(F) = \sum_i f_i \otimes \ell_i$, where:

$$f_i(h) = F(h \cdot e_i), \quad \ell(v) = \langle v, e_i \rangle$$

So Ψ is also linear and injective, and one can see that it does not depend on the basis (similar to the previous question). Clearly, $\Psi \circ \Phi = id$ and $\Phi \circ \Psi = id$.

Exercise 8. *Let F be a local field, $F \neq \mathbb{R}, \mathbb{C}$, and let V be a vector space over F. Show that $h(V) \cong D(V)$.*

Solution: Recall that we defined $D(V) = |\Omega^{top}(V)|$, so for every Haar measure we wish to define a bilinear form $\omega \in \Omega^{top}(V)$ such that $\omega(Av_1, \dots, Av_n) = |\det A| \cdot \omega(v_1, \dots, v_n)$. Given a Haar measure h we define:

$$\omega_h(v_1, \dots, v_n) = h\left(\{\alpha_1 v_1 + \cdots + \alpha_n v_n : \|\alpha_i\| \leq 1\}\right)$$

We need to show the property above for every matrix A, i.e.:

$$h\left(\{\alpha_1 Av_1 + \cdots + \alpha_n Av_n : \|\alpha_i\| \leq 1\}\right) = |\det A| \cdot h\left(\{\alpha_1 v_1 + \cdots + \alpha_n v_n : \|\alpha_i\| \leq 1\}\right)$$

when the norm is of course the p-adic norm. By Gauss decomposition it is enough to show for A diagonal, and matrices of the form: $I + aE_{ij}$ (and of course we can assume $v_i = e_i$). Start with diagonal matrices - $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. As mentioned in class, the Haar measures on V are the product measures of the Haar measures on F. In addition we defined the absolute value on p-adics:

$|k| := \mu(kS)/\mu(S)$. Denote by $S = \{\alpha_1 e_1 + \cdots + \alpha_n e_n : \|\alpha_i\| \leq 1\} = S_1 \times \cdots \times S_n$, $S_i \subset F$. Therefore we get:

$$\begin{aligned} h_n\left(\{\alpha_1\lambda_1 e_1 + \cdots + \alpha_n\lambda_n e_n : \|\alpha_i\| \leq 1\}\right) &= h_1(\lambda_1 S_1) \times \cdots \times h_n(\lambda_n S_n) \\ &= |\lambda_1| h_1(S_1) \times \cdots \times |\lambda_n| h_n(S_n) \\ &= |\lambda_1 \dots \lambda_n| h(S) \\ &= |\det A| h(S) \end{aligned}$$

Now consider matrices of the form $A = I + aE_{ij}$, for some $a \in F$. In this case we get:

$$\begin{aligned} &h\left(\{\alpha_1 A e_1 + \cdots + \alpha_n A e_n : \|\alpha_i\| \leq 1\}\right) = \\ &h\left(\{\alpha_1 e_1 + \cdots + (\alpha_i + a\alpha_j) e_i + \cdots + \alpha_n e_n : \|\alpha_i\| \leq 1\}\right) \\ &:= h(\tilde{S}) \end{aligned}$$

Notice that if $\|a\| \leq 1$ then since $\|\alpha_i + a\alpha_j\| = \max(\|\alpha_i\|, \|a\alpha_j\|)$ we get that when $\|\alpha_j\| \leq 1$, it holds that $\|\alpha_i\| \leq 1 \iff \|\alpha_i + a\alpha_j\| \leq 1$. So in this case (under the assumption of $\|a\| \leq 1$) we get the set equality $S = \tilde{S}$, i.e.:

$$\{\alpha_1 e_1 + \cdots + (\alpha_i + a\alpha_j) e_i + \cdots + \alpha_n e_n : \|\alpha_i\| \leq 1\} = \{\alpha_1 v_1 + \cdots + \alpha_n v_n : \|\alpha_i\| \leq 1\}$$

In case where $\|a\| > 1$, we can split $\tilde{S}$ to a finite number of sets such that $\|a\alpha_j\| \leq 1$: Given $\alpha_j \in B_{0,1}$ we present the unit ball as a union of ϵ-balls: $B_{0,1} = \cup_k B_{k,\epsilon}$, for $\epsilon < 1/\|a\|$. This gives a split of the set $\tilde{S}$: $\tilde{S} = \cup_k \tilde{S}_k$, where:

$$\tilde{S}_k = \{\alpha_1 e_1 + \cdots + (\alpha_i + a\alpha_j) e_i + \cdots + \alpha_n e_n : \alpha_j \in B_{i,\epsilon} \wedge \forall l \neq j,\ \|\alpha_l\| \leq 1\}$$

By translation we can center the balls around zero and get a set of the same measure:

$$N_k = \{\alpha_1 e_1 + \cdots + (\alpha_i + a\alpha_j) e_i + \cdots + \alpha_n e_n : \alpha_j \in B_{0,\epsilon} \wedge \forall l \neq j,\ \|\alpha_l\| \leq 1\}$$

$$h(N_k) = h(\tilde{S}_k)$$

Now, as in N_k, $\|\alpha_j\| \leq \epsilon$, we again get that $\|\alpha_i\| \leq 1 \iff \|\alpha_i + a\alpha_j\| \leq 1$ and so $h(N_k) = h(S_k)$ (where S_k is the corresponding split of S). So:

$$h(\tilde{S}) = \sum h(\tilde{S}_k) = \sum h(N_k) = \sum h(S_k) = h(S)$$

Exercise 9. *Show:*

$$S(V, h_V) \cong \left\{ \xi \in S^*(V) \middle| \begin{array}{c} supp(\xi) \textit{ is compact and } \exists \textit{ open compact subgroup} \\ K \subset V \textit{ such that } \forall k \in K,\ k\xi = \xi \end{array} \right\}$$

Solution: $\supseteq$: Clearly every $\xi \in$RHS is compactly supported. For every $x \in V$, Kx is an open neighborhood of x, and $\xi(Kx) = K\xi(x) = \xi(x)$, meaning ξ is locally constant, so we can identify ξ with some $f \in S(V, h_V)$.
$\subseteq$: Take $f \in S(V, h_V)$, then for every $x \in V$ there exists an open group G_x that stabilize f in an open neighborhood U_x of x (since f is locally constant). These neighborhoods cover *supp*(f) (which is compact) so there exists a finite sub-cover: $supp(f) \subset \cup_{i=1}^{n} U_{x_i}$. Consider the finite intersection:

$$G = \bigcap_{i=1}^{n} G_{x_i}$$

It is an open (and thus non-empty) group, that stabilize f everywhere. Thus we can identify f with $\xi \in$RHS, with $K = G$.

Exercise* 10. *1. Find (M, C^∞) which is "almost" a smooth manifold, but is not Hausdorff.*

2. Find (M, C^∞) which is "almost" a smooth manifold, but is not para-compact.

Solution:

1. Consider the line with two origins: $M = \mathbb{R} \times \{a\} \cup \mathbb{R} \times \{b\} / \sim$, with the equivalence: $\forall x \neq 0,\ (x, a) \sim (x, b)$. Clearly M is second countable and has a smooth atlas, but is not Hausdorff.

2. Consider the long open ray: $M = \omega_1 \times [0, 1) \setminus (0, 0)$. We have a topology with respect to the order on M. Clearly M is Hausdorff, and has a smooth atlas, but it is not para-compact:
Start with the open covering $[0, \alpha)$ for every ordinal $\alpha < \omega_1$. Let X be some refinement of this covering, and let S be the set of limit ordinals below ω_1. Then S is a stationary subset of ω_1 (i.e., it intersects every club set in ω_1). For each $\beta \in S$, we pick a $Y_\beta \in X$ such that that $\beta \in Y_\beta$. Consider the function $f : S \to \omega_1$ which sends β to the least ordinal in Y_β. For any $\beta \in S$, $f(\beta) < \beta$ since β is a limit ordinal. So by Fodor's Lemma, There exists a stationary subset $S' \subset S$, such that f is constant on S'. Let γ be the value of f on S', then:

$$\forall \beta \in S', \quad \gamma \in Y_\beta$$

Thus the refinement X is not finite (or even countable) at γ.

Exercise 11. *Let $E = \left\{(\theta, x) : \theta \in S^1,\ x \in L_{\theta/2}\right\}$, where $L_{\theta/2}$ is a line in direction $\theta/2$. Define a structure of a vector bundle on E and show that it is not homeomorphic to the cylinder.*

Solution: We show that E has a structure of a vector bundle over S^1. Clearly they are both topological spaces with the continuous projection:

$$\pi(\theta, x) = \theta \in S^1$$

In addition, for every $\theta \in S^1$, $\pi^{-1}(\theta) = L_{\theta/2}$ which has a structure of a finite-dimensional real vector space. Finally, take any $\theta \in S^1$, let $U \subset S^1$ be an open half circle, such that $\theta \in U$ and consider $\pi^{-1}(U) = \left\{(\theta, x) : \theta \in U,\ x \in L_{\theta/2}\right\}$. It is clearly homeomorphic to $\mathbb{R}^2$, and so E is indeed a vector bundle over S^1. It is obviously not homeomorphic to the trivial vector bundle over S^1, as it is not orientable(and the trivial is). Note that E is homeomorphic to the Mobius ring - by rescaling the fibers.