Wave front Set

Wave front Set is an object related to distributions, which reflects in which directions the distribution is singular.
While we shall talk about it's definition later on, it's characterizing properties shall be discussed now.
Let F be a field, and let M be an analytical manifold.
$\xi \in C^{-\infty}(M)$
$WF(\xi) \subset T^*(M)$ - WF is the Wavefront set.

WF properties

1. $\overline{WF(\xi)} = WF(\xi)$ - the wave front set is a closed set
2. $(x,v) \in WF(\xi) \underset{\forall \lambda \in F}{\Longrightarrow} (x, \lambda v) \in WF(\xi)$ - the wave front set is a conic with respect to multiplication by scalars in the fibers of $T^*(M)$
3. $P(WF(\xi)) = WF\{\xi) \cap M = supp(\xi)$ where $P : T^*M \to M$
4. ξ - smooth $\Leftrightarrow WF(\xi) \subset M$
5. $WF(f\xi_1 + g\xi_2) \subset WF(\xi_1) \cup WF(\xi_2)$ where $f, g \in C^\infty(M)$ or $f, g \in C^\infty(M, Hom(E_1, E_2))$
   Before presenting the rest of the properties, let us recall:
   A **pull back** of function $f : Y \to \mathbb{C}$ by π is defined as$\pi^*(f) = f \circ \pi$

$$\begin{array}{cccc} \pi: & X & \longrightarrow & Y \\ & \pi^*(f) & \searrow & \downarrow f \\ & & & \mathbb{C} \end{array}$$

   We define the **push forward of a distribution** to be
   $< \pi_*(\xi), f >=< \xi, \pi^*(f) >$
   a **pull-back of a distribution:**

6. Having ξ - a distribution on Y, and a submersion π is a smooth function$\pi : X \to Y$,a pullback of ξ along π can be seen as $\xi \circ \pi$, or
   $< \pi^*(\xi), f >=< \xi, \pi_*(f) >$ where $f \in C^\infty(Y)$
7. let $\pi : X \to Y$ be a submersion , $\xi \in C^{-\infty}(Y)$
   $WF(\pi^*(\xi)) = \pi^*(WF(\xi)) = \{(x,v) \in T^*(X) | \exists (y,w) \in WF(\xi)\ s.t.\ \pi(x) = y\ and\ d^*_X \pi(w) = v\}$
   where $d^*_X$is a co-differential.
8. For $\pi : X \to Y$,$\xi \in C^{-\infty}(Y)$, $\pi|_{supp\xi}$-proper map (the inverse map of compact subset is compact)
   $WF(\pi_*(\xi)) \subset \pi_*(WF(\xi)) = \{(y,w) \in T^*(Y) | \exists\, (x,v) \in WF(\xi)\ s.t.\ \pi(x) = y,\ d^*_X \pi(w) = v\}$
   In order to proove the uniqueness of the wavefront, we should consider the following:

The WF is defined uniquly on $\mathbb{R}$ by axioms 1-4.
For a linear space V and a function $p \in V*$, take a cutoff functionρ - which takes only a small area into account.
By characteristic 7 - (*)$WF(p_*(\rho\xi)) \subset p_*(WF(\rho\xi)) \subset \mathbb{R}$ which means $p_*(\rho\xi)$ is a smooth function. This means that the vector$p$ is in the wavefront, but not only direction p is relevant, but also nearby directions.
(*) sets the lower bound. In order to get the upper bound, we need to look at ξ as a push forward of another function. We shall use the Radon transform to do that: with L the space of lines, $\mathbb{C} \to V$, $\mathbb{C} \to L$, and $Radon : V \to L$, $Radon^{-1:} L \to V$
The idea now is to show that the Radon transform turnes the inclusion of character 7 into equality, and then, with equality from both sides (on characters 6 and 7) we get uniqueness.

EX^*: Prove that the 7 characteristics define the WF uniquely.

We shall prove existance in the general case, in order to cover both real and p-adic cases (Hormander defined through Fourier transform, which checks how much the function is smooth over the real space).

Def : for $\xi \in C^{-\infty}(V)$,
$(0,v) \notin WF(\xi) \Leftrightarrow \exists f \in C^{-\infty}(V),\ f(0) \neq 0,\ \exists U \ni v\ s.t.\ \left(U \times \mathbb{R} \ni (v', \lambda) \mapsto \widehat{f\xi}(\lambda v')\right) \in S^{\mathbb{R}}(U \times \mathbb{R})$
where $S^{\mathbb{R}}(U \times \mathbb{R})$ are the Schwartz functions along $\mathbb{R}$.
In books you may found the definition for the real case $(0,v) \notin WF(\xi) \Leftrightarrow \forall n \exists c\ s.t.\ \widehat{f\xi}(\lambda v') < C_n \left(\frac{1}{1+\|\lambda\|}\right)^n$ and that helps defining a smooth distribution in a certain direction.

In light of the definition, prooving charateristics 1-5 are easy, and 7 is easy given the 6'th characteristic:

$$X \xrightarrow{f} Y$$

$$x \mapsto (x, f(x))$$

$$X \overset{embedding}{\longrightarrow} X \times Y$$

$$\searrow \quad \downarrow projection$$

$$Y$$

In order to proove characteristic 6, it is needed to understand why the **change of variables is valid** (the proofs of the p-adic case can be found in Heifetz's work. The relevant part in Hormander's book is in chapter 8)
We'll proove now that the definition is relevant to diffeomorphic variables change: Because $\xi \in C^{-\infty}$ acts "nicely" on several directions, we would like to check that its transform acts the same on those directions. On those directions the differential of the veriables change function operates on ξ.
Take$\varphi : V \to V$, where $Id = D_0\varphi : V \to V$, then:

$$\widehat{\rho\varphi^*(\xi)} \sim \widehat{\rho\xi}$$

By the characteristics, we can calculate:
for

$$\mu \in C^{\infty}(M, D_M),\ i : M \hookrightarrow N, WF(i_*(\mu)) = CN_N^M$$

where $CN_N^M \subset T^*M$ is the co-normal bundle, where $N \subset M$
Ex : Calculate:

1. $WF(\delta)$
2. $WF(\sin)$
3. $WF(X_+^{\lambda})$ (from the first lesson)
4. $WF(\Delta w)$
5. $WF(i_*(\mu))$ can be calculated manually, but it is difficult. Use characteristics

In the beggining of the lesson, we defined push-forward in a spesific way and developed it. We did not do the same for pull-back.
Define a pull-back for a smooth distribution regarding some mapping.
$C_\Gamma^{-\infty}(M) = \{\xi | WF(\xi) \subset \Gamma\}$, $\Gamma \subset T^*M$. We have convergence such that

$$C_\Gamma^{-\infty}(M) \ni \xi_i \underset{\Gamma}{\to} \xi \in C_\Gamma^{-\infty}(M)$$

for $\pi : X \to Y$ we'll take $S_\pi \subset T^*Y$, $S_\pi = \{(y,v) | \exists x\ s.t.\ \pi(x) = y,\ d_X^*(v) = 0\}$where d_X^* is the co-differential. S_π gives the zero section on the image, and actually "checks" how much the function is not an immersion.

$$Thm: Y \subset \Gamma \subset T^*(Y),\ \Gamma \cap S_\pi \subset Y, \qquad \begin{array}{cccc} \pi^*: & C^\infty(Y) & \to & C^\infty(X) \\ & \cap & & \cap \\ & C_\Gamma^{-\infty}(Y) & \to & C_\Gamma^{-\infty}(X) \end{array}$$

Ex : Prove the theorem

To show uniqueness, one should show that $C^\infty(Y)$ is dense in $C_\Gamma^{-\infty}(Y)$. In order to do so, take $C_\Gamma^{-\infty}(Y) \ni \rho_i \to \delta \in C^\infty(Y)$ and define $\xi_i := \rho_i * \xi$ and that approximates ξ into a smooth function, then define $\pi^*(\xi) := lim \pi^*(\xi_i)$ and its convergence is all that is left to prove.

ξ_1, ξ_2 $WF(\xi_1) \cap WF(\xi_2) \subset M$ but we do not know how to call $\xi_1 \cdot \xi_2$. Therefore, we shall define $\xi_1 \boxtimes \xi_2 \in C^{-\infty}(M \times M)$ ($\boxtimes$is called outer product). $C_c^\infty(M) \otimes C_c^\infty(M) \to C_c^\infty(M \times M)$

In order to define the inner multiplication, we shall use an outer one and pull it back through the diagonal:

$M \hookrightarrow M * M$

$x \mapsto (x, x)$

This will work due to the fact that $T^*(x * y) = T^*(x) * T^*(y)$ and this equation shows that the WF of an outer multiplication is equal to the multiplication of WF's.