Algebraic Topology HW1

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NOTE: The number in each exercise corresponds to the number of the exercise on the sheet.

Problem 2

Let $f : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ such that $x \mapsto \left(\frac{x}{||x||}\right)$, which maps each point to its corresponding unit vector in S^{n-1}. Clearly f is defined since we can take the point $x = 0$ out of the plane and this map would work. And let $g : S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ such that $x \mapsto x$, as the identity map, since $S^{n-1} \subset \mathbb{R}^n \backslash \{0\}$. Here x represents the vector $(x_1, ..., x_n)$. By this we can see how $f \circ g = Id_{S^{n-1}}$ and $g \circ f \equiv f$. Thus we can construct a map $H(x,t) : \mathbb{R}^n \backslash \{0\} \times I \to \mathbb{R}^n \backslash \{0\}$ such that

$$H(x,t) = \frac{x}{1 + (||x|| - 1)t} \tag{0.1}$$

Clearly, $H(x,0) = x = Id_{\mathbb{R}^n \backslash \{0\}}$ and $H(x,1) = \frac{x}{||x||} \equiv f$. Since H is clearly continuous, we have shown that $\mathbb{R}^n \backslash \{0\}$ is homotopically equivalent so S^{n-1}. Let us see it in terms of a graph in $\mathbb{R}^2$ in Figure 1.

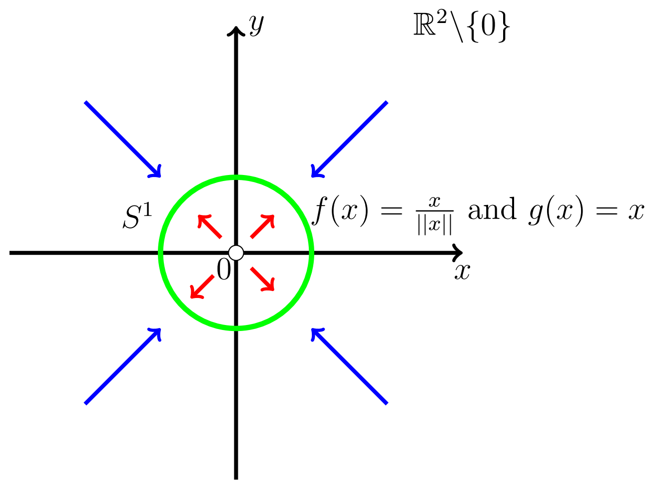

Figure 1: This figure shows the equivalence between the spaces.

We can clearly see how the homotopy equivalence is made by expanding through f the points that lay inside the unit circle and by contracting the points outside of the unit circle. This can be done since the zero point is empty. On the other hand, every point in the unit circle, goes to itself by g.

Problem 4

In order to show that $T^2\backslash\{*\}$ is homotopically equivalent to $\mathbb{R}^2\backslash\{*_1, *_2\}$, we will use the transitivity property of the homotopy relation and show that $T^2\backslash\{*\} \sim S^1 \vee S^1 \sim \mathbb{R}^2\backslash\{*_1, *_2\}$. First of all, we identify the torus with the filled square that has equivalent parallel sides and just take away a point inside it, as we can see from Figure 2. Furthermore, we identify $S^1 \vee S^1$ with the unfilled square, with parallel sides equivalent to each other. This can be done, by making all the corners of the unfilled square be equivalent: this will be the point where the circles are glued together in the bouquet.

So let $g : T^2\backslash\{*\} \to S^1 \vee S^1$ such that g maps $x \in T^2\backslash\{*\}$ by projecting it into the boundary of the square as defined by the intersection of the boundary with the straight line passing through x and the centre of the square (which is the missing point of the torus). This is a well defined map since there is a natural direction of projection, which makes all points x in the same line equivalent under $g(x)$ and the centre of the square is not mapped to any point as it is not in the domain. On the other hand, $f : S^1 \vee S^1 \to T^2\backslash\{*\}$ is defined by the inclusion map, so $f(x \in S^1 \vee S^1) = x \in T^2\backslash\{*\}$.

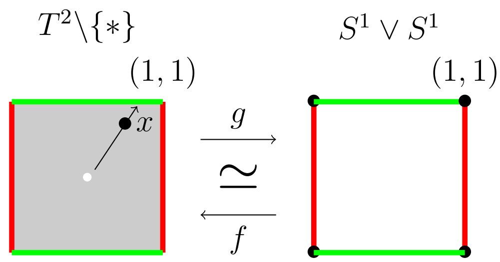

Figure 2: The map g maps the torus without a point to the bouquet of two circles and f goes in the backward direction. As we can see from the diagram, g maps each point in the torus to a point in its boundary, which is the bouquet of two circles. There is a homotopy between the identity map and the composition of the two maps, as shown in the exercise, $f \circ g \sim Id_{T^2\backslash\{*\}}$ and $g \circ f \sim Id_{S^1 \vee S^1}$

Thus now we can show the homotopy map, $H(x,t) = x + t(g(x) - x)$. At $t = 0$, $H(x,0) = x = Id$ and at $t = 1$, $H(x,1) = g(x) = f \circ g(x)$. Clearly, since H is continuous, the two spaces are homotopically equivalent.

Now, let us prove that $S^1 \vee S^1 \simeq \mathbb{R}^2\backslash\{*_1, *_2\}$. Define $\varphi : S^1 \vee S^1 \to \mathbb{R}^2\backslash\{*_1, *_2\}$ as an embedding, where the glueing point in the bouquet becomes the origin $(0,0)$ on the plane, as seen in Figure 3. So each point in the bouquet goes to its self representative in $\mathbb{R}^2\backslash\{*_1, *_2\}$, where we let the missing points be the centres of the circles respectively. On the other hand, define $\gamma : \mathbb{R}^2\backslash\{*_1, *_2\} \to S^1 \vee S^1$ by letting each point, x, on the plane be mapped to the first intersection point, y, between any of the two circles and the straight line passing through x and the origin $(0,0)$. Similarly as in the first part, all points x laying in the same line, will be mapped to the same $z \in S^1 \vee S^1$, the points outside the circles will move towards the origin and intersect externally with the circles and the points laying inside the circles will move away from the origin and intersect internally the circles. Furthermore, $\gamma(0,y) = \gamma(x,0) = (0,0)$. It is easy to see that this is a well defined map, maybe with some help of Figure 3, and that this map is continuous since it is dependent on the angle of the line from the origin with the x-axis. The map γ is uniquely defined by the origin. We can see how the plane without two points can "shrink" to the bouquet of two circles.

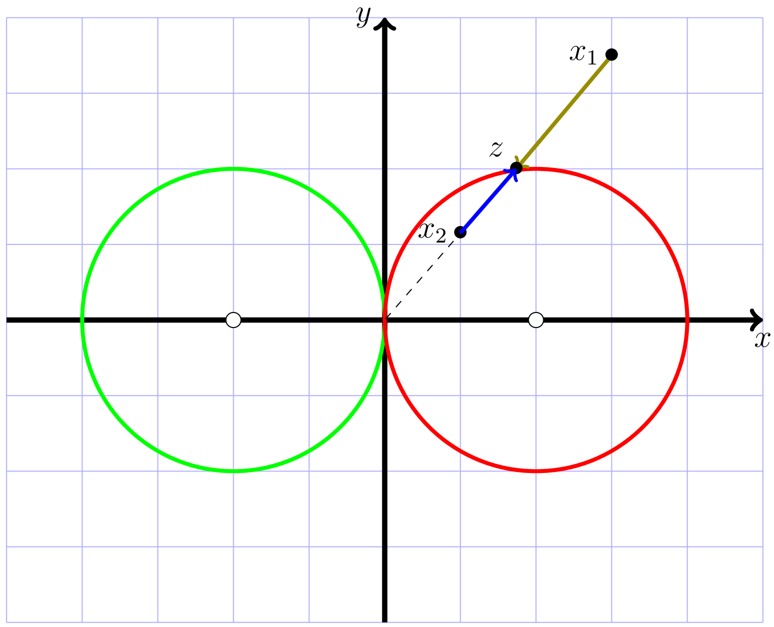

Figure 3: In this figure we can identify, first the embedding of $S^1 \vee S^1$ into $\mathbb{R}^2 \backslash \{*_1, *_2\}$ by φ and also the "deformation" of the plane without two points into $S^1 \vee S^1$ through the γ map. We can clearly see how each point on the same line maps to the same point in the bouquet.

We can define $H(x,t)$ in an analogous way as before and conclude that $S^1 \vee S^1 \simeq \mathbb{R}^2 \backslash \{*_1, *_2\}$, so by transitivity, $T^2 \backslash \{*\} \simeq \mathbb{R}^2 \backslash \{*_1, *_2\}$

Problem 5

First of all, let us prove that a continuous map $f : X \to Y$ preserves paths. Meaning that if x_0 and x_1 are connected by a path, then $f(x_0)$ and $f(x_1)$ are also connected by a path. Indeed, if $\gamma : [0,1] \to X$ is a path so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$, then $f \circ \gamma : [0,1] \to Y$ is also a path such that $f(\gamma(0)) = f(x_0)$ and $f(\gamma(1)) = f(x_1)$, which makes the two points connected.

Let us assume that $X \simeq Y$, and that $\pi_0(X)$ and $\pi_0(Y)$ are the sets of their path connected components, respectively. Let $f : X \to Y$ and $g : Y \to X$, then we write

$\pi_0[f] : \pi_0(X) \to \pi_0(Y)$ and similarly with $\pi_0[g]$. Now, we know that by definition of the map $\pi_0[f]$, if $f(x_0) = y_0 \in Y_0 \in \pi_0(Y)$, where $x_0 \in X_0 \in \pi_0(X)$, then $\pi_0[f](X_0) = Y_0$.

Now, let us start by injectivity of $\pi_0[f]$, let $\pi_0[f](X_1) = \pi_0[f](X_2) = Y_1$, so take $y_1, y_2 \in Y_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$, where by definition $x_1 \in X_1$ and $x_2 \in X_2$. Thus since y_1 and y_2 are connected, being in the same path connected component, the continuity of g tells us that x_1 and x_2 are connected, as we saw in the proof above, making $X_1 = X_2$, so $\pi_0[f]$ is injective.

Now let us look at surjectivity of $\pi_0[f]$. Since we have by definition that $\pi_0[f]([x]) = [f(x)] \in \pi_0(Y)$, where the brackets indicate the path component of x and of $f(x)$. Yet, we know that the map $f \circ g$ is homotopic to the identity, so the path between x_1 and x_2 is homotopic to the path between $f \circ g(x_1)$ and $f \circ g(x_2)$. Thus we must have that $[f \circ g(x)] = [x]$ so that $\pi_0[f] \circ \pi_0[g][x] = [x]$, which implies that $\pi_0[f] \circ \pi_0[g] = Id_{\pi_0(Y)}$, meaning that $\pi_0[f]$ is surjective, where in fact by symmetry we have $\pi_0[g] \circ \pi_0[f] = Id_{\pi_0(X)}$, making $\pi_0[f]$ and $\pi_0[g]$ mutually inverse bijections.

It is easy to see that this induces a homotopy equivalence on path components since let X_i be a path component of X that maps to Y_i, a path component of Y. Then we have that $f(X_i) \subseteq Y_i$ and $g(Y_i) \subseteq X_i$. Thus the composition of the respective restriction maps $f|_{X_i} \circ g|_{Y_i}$ and $g|_{Y_i} \circ f|_{X_i}$ are homotopic to the identity by the homotopy of $f \circ g$ and $g \circ f$.

Problem 6

$(\Rightarrow)$

Assume that $f \sim g$ are homotopic, where $f : X \to Y$ and $g : X \to Y$ such that $g(x) = y_0$ for some $y_0 \in Y$ and for all $x \in X$. We know that the *cone*$(f) = C_f = [(X \times I) \sqcup Y/(x, 0) \sim f(x)] /X \times \{1\}$. Let $\widetilde{f} : C_f \to Y$ be an extension of f. If we identify $X \hookrightarrow C_f$ by noting that $X \times \{0\} \subset C_f$, we have that $\widetilde{f}|_X = f$, which is continuous over X. Thus now, we need to define the map $\widetilde{f}$ over $C_f \backslash (X, 0)$ by letting $\widetilde{f}(z) = f(x_0)$ where $z \in C_f \backslash (X, 0)$ and x_0 is the base point of X, for all $x \in X$. Thus we clearly have that $\widetilde{f}$ is continuous and well defined and that $\widetilde{f}$ is an extension of f.

$(\Leftarrow)$

Let us assume there exists an extension of f, $\widetilde{f} : C_f \to Y$ such that $\widetilde{f}|_X = f$ an $\widetilde{f}$ is continuous. We thus need to find a homotopy $H(x, t) : X \times I \to Y$ such that $H(x, 0) = f$ and $H(x, 1) = g = y_0$ for some $y_0 \in Y$, i.e.: the constant map. Let us define H in the following way, first define two kinds of points, $\widetilde{f}(x, 0) = \widetilde{f}|_X = f$ and $\widetilde{f}(x, 1) := g(x) = y_0$, then $H : X \times I \to Y$ looks like,

$$H(x, t) := \widetilde{f}([x, t]) \tag{0.2}$$

Where $H(x, 0) = \widetilde{f}([x, 0]) = f(x)$ and $H(x, 1) = \widetilde{f}([x, 1]) = g(x)$, which is clearly continuous by the continuity of $\widetilde{f}$. Thus we have a homotopy between f and the constant map, g.

Problem 7

Let $f : X \to Y$, then we can construct the mapping cylinder, $M_f = (X \times I) \sqcup Y/(x,0) \sim f(x)$. Clearly we can identify $Y \subset M_f$ by seeing $y \in Y$ as $y \in M_f$, thus it is natural to define two maps in the following way,

$$\begin{aligned} \varphi : M_f &\to Y \\ (x,s) &\mapsto f(x) \\ y &\mapsto y \end{aligned} \tag{0.3}$$

$$\begin{aligned} \gamma : Y &\to M_f \\ y &\mapsto y \end{aligned} \tag{0.4}$$

Now that we have these two maps, which are continuous, let us define a homotopy between their composition and the identity map. Write $x_s = (x,s) \in M_f$ and let $H(x_s,t) : M_f \times I \to M_f$, where $H(z,t) = z + t(\varphi(z) - z)$, we can see that $H(z,0) = z \equiv Id_{M_f}(z)$ and $H(z,1) = \varphi(z) \equiv \gamma \circ \varphi(z)$, where $z \in M_f$. Thus we have just shown that $M_f \simeq Y$.

Problem 8

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Let s_0 be the special point in the pointed space S^1. We know that $\Sigma(X) = [cone(X)/X \times \{1\}]/\{x_0\} \times I$ which yields a suspension of X through the interval I, where the line passing through the end points $(x_0,1)$, $(x_0,0)$ and the special point $x_0 \in X$ is collapsed into one point. We write that $[(x,0)] = [(x,1)] = [(x_0,t)]$ for $t \in I$. And on the other hand $S^1 \wedge X = S^1 \times X/[S^1 \sqcup X/\{s_0 \sim x_0\}]$ where we denote $[S^1 \sqcup X/\{s_0 \sim x_0\}] = S^1 \vee X = [A] \in S^1 \wedge X$.

Now we define $f : \Sigma(X) \to S^1 \wedge X$ and $g : S^1 \wedge X \to \Sigma(X)$ such that,

$$f = \begin{cases} f(x,t) = [(t,x)] & \text{for } x \neq x_0, t \neq 0,1 \\ f([x,t]) = [A] & \text{else} \end{cases} \tag{0.5}$$

$$g = \begin{cases} g([A]) = [(x,0)] & \text{for } (s,x) = (s_0,x_0) = [A] \\ g(s,x) = [(x,s)] & \text{for } s \neq s_0, x \neq x_0 \end{cases} \tag{0.6}$$

Now, since each equivalence class $[(x,t)] \in \Sigma(X)$ such that $x \neq x_0, t \neq 0,1$ is a singleton, and similarly for $[(s,x)] \in S^1 \wedge X$ such that $s \neq s_0, x \neq x_0$. This is continuous even if it is defined piecewise since the points near the singleton map to the points near the other singleton. Thus we can clearly see how these two maps are self inverses and there is a bijection between them. Since they are continuous, we can conclude that the two spaces are homeomorphic, $\Sigma(X) \cong S^1 \wedge X$.