ALGEBRAIC TOPOLOGY-EXC 2

ITAY GLAZER

Q1) Show that:

a) $\pi_0 \circ \Omega = \pi_1$.

b) $\Omega(\Omega(X)) = Mor(S^2, X)$.

Proof. a) Let (X, x_0) be a pointed topological space. $\Omega(X, x_0) = cont((S^1, s_0), (X, x_0), \{f_0\})$ when $f_0(S^1) = x_0$. We want to prove that $f, g \in \Omega(X, x_0)$ are path connected in in$\Omega(X, x_0)$ iff they are homotopic in (X, x_0),i.e $[f] = [g] \in \pi_1(X, x_0)$.

There is an homeomorphism between $cont(X \times Y, Z)$ to $cont(X, cont(Y, Z)$ (if Y locally compact Hausdorff and X is Hausdorff) by $f \longmapsto (x \longmapsto f_x)$ when $f_x(y) = f(x, y)$. So for any $f_1, f_2 : S^1 \longrightarrow X$ such that $f_1 \sim f_2$, there is an homotopy $H : I \times S^1 \longrightarrow X$. This homotpoy correponds to the path $\gamma : I \longrightarrow cont(S^1, X)$ by $\gamma(t)(s) = H(t, s)$, when $\gamma(0) = f_1$ to $\gamma(1) = f_2$. We do the same for the other direction and we get that $\pi_0 \circ \Omega = \pi_1$.

b)

$$\Omega(\Omega(X)) = \Omega(cont(S^1, X)) = cont(S^1, cont(S^1, X))$$

And we showedin class that

$$cont(\Sigma X, Z) \cong cont(X, \Omega(Z))$$

So we have:

$$\Omega(\Omega(X)) = cont(S^1, \Omega(X)) = cont(\Sigma S^1, X) = cont(S^2, X)$$

Since

$$\Sigma S^1 = S^1 \times I / \left(\left(S^1, \{0\}\right) \sim (s_0, \{0\}), \left(S^1, \{1\}\right) \sim (s_0, \{1\}) \right), (s_0 \times I) \sim (point)$$

And this equals S^2/p where p is gluing the arc between the north and the south pole into one point and it is homeomorphic to a sphere. □

Q2) (Hatcher Chapter 1.1, Ex. 5) Show that for a space X TFAE:

a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Proof. *a)* $\Longrightarrow$ *b)*. Let $f : S^1 \longrightarrow X$. We define $\widetilde{f} : D \longrightarrow X$ as follows $\widetilde{f}(x) = H(\frac{x}{\|x\|}, 1 - \|x\|)$ for $\|x\| > 0$ and $\widetilde{f}(0) = x_0$ when H is the homotopy from f to a constant map, i.e $H(x, 0) = f(x)$ and $H(x, 1) = x_0$. Notice that $\widetilde{f}|_{S^1} = f$. Also $\widetilde{f} = H \circ g$ when $g : D/\{0\} \longrightarrow S^1 \times I$ by $x \longmapsto (\frac{x}{\|x\|}, 1 - \|x\|)$ and g, H continous at $x \neq 0$ so we only need to prove that $\widetilde{f}$ continous at 0. Let V be an open neighborhood of x_0 in X. Then $H^{-1}(V)$ is open in $S^1 \times I$ and contains $S^1 \times \{1\}$, and hence also $S^1 \times (1-\epsilon, 1]$ (for any $s \in S^1$ there is $U_s \subseteq S^1$ and $V_{1,s} \subseteq I$ such that $U_s \times V_{1,s}$covers $(\{s\}, \{1\})$, and since $S^1$is compact we can cover $S^1 \times \{1\}$ by finite such sets get that $S^1 \times \bigcap_{i=1}^{n} V_{1,s_i}$is an open neighborhood of $S^1 \times \{1\}$ so it contains some open neighborhood $S^1 \times (1 - \epsilon, 1]$ of it).

so:

$$\widetilde{f}^{-1}(V) = g^{-1} \circ H^{-1}(V) \cup \{0\} \supseteq g^{-1}(S^1 \times (1 - \epsilon, 1]) \cup \cup\{0\} \supseteq B_\epsilon(0)$$

So $\widetilde{f}$ continous at 0 and hence is continous in all D.

b) $\Longrightarrow$ *c)* Let $f : S^1 \longrightarrow X$, $f(s_0) = x_0$ So it can be extended to $\widetilde{f} : D \longrightarrow X$. We can define the following homotopy $H_1(\overrightarrow{s}, t) = \widetilde{f}(\overrightarrow{s}(1 - t))$. So $H_1(s, 0) = \widetilde{f}(s) = f(s)$ and $H_1(s, 1) = \widetilde{f}(0) = y_0$. But this homotopy is not "good" since its not always holds that $H_1(s_0, t) = x_0$. Lets try to fix it:

For convenience, we can identify S^1 with the interval $[0, 1]$ and demand that $H_1(1, t) = H_1(0, t)$ so we can write $H_1 : I \times I \longrightarrow X$.

Notice that have a path $\gamma(t) = H_1(0, t)$ (or $H_1(1, t)$) . We creat an homotopy H that at time t has:

- $H(s, t) = \gamma(3st)$ for $s \leq 1/3$

-$H(s, t) = H_1(3s - 1, t)$ for $1/3 \leq s \leq 2/3$.

- $H(s, t) = \gamma((3 - 3s)t)$ for $2/3 \leq s \leq 1$.

So $H(s, t)$ at time t , starts in x_0 in $s = 0$, do the path from $x_0$to $\gamma(t)$ until $s = 1/3$ (3 times faster) and then do $H_1(s, t)$ (3 times faster) and goes back to x_0.

Its easy to see that $H(s_0, t) = x_0$. If we prove that H is indeed an homotopy, then we get:

$H(s,1)$ is tha path from $x_0$to y_0 and then back to x_0 in the same way, so $H(s,1) \sim const(x_0)$ and $H(s,0)$ is the map that start from $x_0$stays in place until $s = 1/3$ and then do the path $H_1(s,0) = \widetilde{f}(s) = f(s)$ 3 times faster and then stay in place until $s = 1$, so by shrinking to waiting time continuously we get that $H(s,0) \sim H_1(s,0) = \widetilde{f}(s) = f(s)$. And we get that $f(s) \sim const$ so $\pi_1(X, x_0) = 0$.

It is left to show that H is continous:

It is easy to see that $H|_{[0,1/3]\times I}$, $H|_{[1/3,2/3]\times I}$, $H|_{[2/3,1]\times I}$are continous in their domain since they are restrictions of continous functions, so we need to show continuity only on the gluing points. Let $(s,t) = \{1/3, t\}$. If we take a neighborhood V of $H(s,t)$ then , $H^{-1}(V) \cap [0,1/3] \times I$ is open and $H^{-1}(V) \cap [1/3, 2/3] \times I$ is open. so they contains balls $[1/3, 1/3+\epsilon) \times B_\epsilon(t)$, and $[1/3, 1/3-\delta) \times B_\delta(t)$ so if we take $m = min(\epsilon, \delta)$ then we have $B_m(1/3) \times B_m(t) \subseteq H^{-1}(V)$ and H is continous.

c) $\Longrightarrow$ *a)* $\pi_1(X, x_0) = 0$ for all $x_0 \in X$ so for every map $f : S^1 \longrightarrow X$ by definition $f \sim const(x_0)$. □

Q3) (Hatcher Chapter 1.1 Ex 16 revised):

a) Let $i : A \hookrightarrow X$ be the inclusion, prove that if A is a retract of X then the induced homomorphismi^* : $\pi_1(A) \rightarrow \pi_1(X)$ is injective.

Proof. If there exists $r : X \longrightarrow A$ such that $r|_A = id|_A$then we have that $r \circ i = id_A$ so $(r \circ i)^*$: $\pi_1(A) \longrightarrow \pi_1(A)$ is the identity. So if $i^*$is not injective then there exists $g \neq 1$ such that $i^*(g) = 1$ so $g = (r \circ i)^*(g) = r^* \circ i^*(g) = 1 \neq g$. Contradiction. So $i^*$is injective. □

b) Show that there are no retractions $r : X \rightarrow A$ in the following cases:

i. $X = R^3$ with any subspace A homeomorphic to S^1 .

ii. $X = S^1 \times D^2$ with $A = S^1 \times S^1$ its boundary torus.

Proof. i) If there is a retraction , than we have in injection from $\pi_1(A) \cong \pi_1(S^1) = Z$ to the trivial group which is imposible.

ii) Again, we have an injection from $\pi_1(A) = Z \times Z$ to $\pi_1(S^1 \times D^2)$. We will prove that $\pi_1(S^1 \times D^2) = Z$ and we will reach a contradiction since if there exists an injection $i : Z \times Z \longrightarrow Z$ then $i(1,0) = m$ and $i(0,1) = n$ and we have that $i(n,0) = mn = i(0,m)$ so its not injective.

Lets calculate $\pi_1(S^1 \times D^2)$: At first, a map $f : Z \longrightarrow X \times Y$ is continous iff the maps f_1, f_2 are both continous, when $f = (f_1, f_2)$. So also, a map $H : Z \times I \longrightarrow X \times Y$ is continous iff H_1, H_2 are continuous when $H = H_1, H_2$. So we have a bijection $\pi_1(S^1 \times D^2, (s_0, d_0)) = \pi_1(S^1, s_0) \times \pi_1(D^2, d_0)$ and since all the spaces are path connected we have $\pi_1(S^1 \times D^2) = \pi_1(S^1) \times \pi_1(D^2) = Z$ □

Q4) Given a topological space X, define a topological monoid $\Omega'(X)$ which is homotopically equivalent to $\Omega(X)$ and a map that sends the product of $\Omega'(X)$ to the non-associative product on $\Omega(X)$.

Proof. The intuition for fixing the non-associativity is as follows:

* We look at the space $\Omega'(X) = \{(\gamma, r)\}$ when $\gamma : S^1 \longrightarrow X$ is a loop and $r \in R^+$. We can think of the elements in $\Omega'(X)$ as loops $\gamma' : [0, r] \longrightarrow X$ s.t $\gamma'(0) = x_0$ and $\gamma'(1) = x_0$. We give it the topology of $\Omega(X) \times R$ (so a sub basis of open sets consists of $U_{K,V} \times (a, b)$).

* We give it a monoid structure by the action $(\gamma, r) * (\beta, t) = (\gamma * \beta, r + t)$, when $\gamma * \beta(t) = \gamma(t)$ for $0 \leq t \leq r$ and $\gamma * \beta(t) = \beta(t)$ for $r \leq t \leq r + t$. So it is easy to see that this operation is associative.

* The identity element will be $(\{x_0\}, 0)$ so $(\{x_0\}, 0) * (\gamma, r) = (\{x_0\} * \gamma, r) = (\gamma, r)$ (the same for right multiplication).

*Now we have to check that this operation $* : \Omega'(X) \times \Omega'(X) \longrightarrow \Omega'(X)$ is continuous. Let $U_{K,V} \times (a, b) \subseteq \Omega(X) \times R$ and look at $*^{-1}(U_{K,V} \times (a, b))$ and lets pick some $(\gamma_1, r_1), (\gamma_2, r_2) \in *^{-1}(U_{K,V} \times (a, b))$. Then we know that $\gamma_1 * \gamma_2(K) \subseteq V$ (when we look at $\gamma_1 * \gamma_2$ as a loop in [0,1] , i.e, shrunk by $r_1 + r_2$, so in $[0, \frac{r_1}{r_1+r_2}]$ we have $\gamma_1(\frac{r_1+r_2}{r_1} t)$ and in $[\frac{r_1}{r_1+r_2}, 1]$ we have $\gamma_2(\frac{r_1+r_2}{r_2}(t - \frac{r_1}{r_1+r_2}))$ and that $r_1 + r_2 \in (a, b)$.

We denote $K_1 := K \cap [0, \frac{r_1}{r_1+r_2}]$ and $K_2 := K \cap [\frac{r_1}{r_1+r_2}, 1]$. Also there exists $\epsilon > 0$ such that $B_\epsilon(r_1) + B_\epsilon(r_2) \subseteq (a, b)$ from continuity of addition. Now, If K_1 is empty, then we can look at:

$$V_1 \times V_2 := \left\{ \left(U_{S^1,V} \times B_\epsilon(r_1)\right), \left(U_{\frac{r_1+r_2}{r_2}(K - \frac{r_1}{r_1+r_2}),V} \times B_\epsilon(r_2)\right) \right\}$$

and we get that if we concatenate paths from V_1 and V_2 we get a path $\gamma = \gamma_1 * \gamma_2$ of length $d \in (a, b)$ and $\gamma(K) = \gamma_2(\frac{r_1+r_2}{r_2}(K - \frac{r_1}{r_1+r_2})) \subseteq V$. So $V_1 * V_2 \subseteq U_{K,V} \times (a, b)$ as required.

If K_2 is empty we do the same trick. If $K_1, K_2 \neq \emptyset$, then we look at:

$$V_1 \times V_2 := \left\{ \left(U_{\frac{r_1+r_2}{r_1}K_1,V} \times B_\epsilon(r_1)\right), \left(U_{\frac{r_1+r_2}{r_2}(K_2 - \frac{r_1}{r_1+r_2}),V} \times B_\epsilon(r_2)\right) \right\}$$

And get that $\gamma(K_1) = \gamma_1(\frac{r_1+r_2}{r_1} K_1) \subseteq V$ and that $\gamma(K_2) = \gamma_2(\frac{r_1+r_2}{r_2}(K - \frac{r_1}{r_1+r_2})) \subseteq V$ so $\gamma(K) \subseteq V$ and again $V_1 * V_2 \subseteq U_{K,V} \times (a, b)$ as required. So * is continuous.

*We need also to check that $\Omega'(X) \sim \Omega(X)$. We set $f : \Omega'(X) \longrightarrow \Omega(X)$ by $(\gamma, r) \longmapsto \gamma$. And $g : \Omega(X) \longrightarrow \Omega'(X)$ by $\gamma \longmapsto (\gamma, 1)$. So we get that $f \circ g = id$ and that $g \circ f((\gamma, r)) = (\gamma, 1)$ is just dividing by r so we can create an homotopy $H(((\gamma, r), t) = (\gamma, r - (r-1)t)$ ant it is an homotopy since its continuous in R (and id in $\Omega(X)$).

*Last thing, if we look at the map f as above, then $f((\gamma_1, 1) * (\gamma_2, 1)) = f((\gamma_1, 1)) * f(\gamma_1, 1)$. □

Q5) Show that the torus is covered by a plane and by a cylinder.

Proof. A torus is covered by a plane by the map $p : R^2 \longrightarrow R^2/(x, y) \sim (x, y) + Z^2$. Now, for every $x \in\longrightarrow R^2/(x, y) \sim (x, y) + Z^2$ there exists a small disc $B_{1/4}(x)$ such that $p^{-1}(B_{1/4}(x))$ is a disjoint countable union of discs $B_{1/4}(y_i) \cong B_{1/4}(x)$ so it is a covering space.

For a cylinder we do the same idea only create the map $p : S^1 \times R \longrightarrow \left(S^1 \times [0, 1]/(s, 0) \sim (s, 1)\right) = S^1 \times S^1$ by gluing $(S^1, r) \sim (S^1, r + z) \forall z \in \mathbb{Z}$. Again, by choosing small neighborhood of a point (s, t) of the form $S^1 \times B_{1/2}(t)$ we get that $p^{-1}(S^1 \times B_{1/2}(t))$ is a disjoint union of sets of the form $S^1 \times B_{1/2}(t + z)$, so its a cover map. □