# Algebraic Topology HW 3

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*Thanks to Gil Goffer for her contribution with her drawings.*

## Problem 0

Let us look at $X$, a cover of $n+1$ genus. We can think of this as a tori with one whole in the middle and the other $n$ wholes, surrounding the middle one. In this way we can see that there exists a cover, $\widetilde{X}$, of $kn+1$ genuses. We first let the 'extra' one whole in $\widetilde{X}$ cover the whole in the middle of $X$. Then we can do an easy induction on $k$, to see how it will cover it. Let $k=1$, we get the trivial cover by mapping each outer genus to its respective on in $X$. Let us assume that it is true for $k=m$, then we for $(m+1)n+1$ genuses, the extra one genus is still covering the middle whole in $X$ and the 'new' $n$ genuses will each cover one of the $n$ genuses in $X$.

In order for this cover to be well defined, we need to define an order of covering and we need to show that all the points have a neighbourhood that is covered by a union of disjoint sets that are homeomorphic to itself. Let us call the *leaves* of the cover or the space, the area around a genus whole up to the boundary of the genus in the middle. Then if we start at any leaf in $\widetilde{X}$ and use this one to cover a given leaf in $X$, then as we move to the right, each leaf will cover the next leaf in $X$, so that in a cyclic mode in $X$, we will cover each leaf $m+1$ times. We can see this more easily in the picture below, where each boundary in $\widetilde{X}$ has been colored to match its respective projection in $X$. As we can see in the picture, all the points in $X$ have a neighbourhood that is covered $m+1$ times, even on the boundary of the middle genus. Note that the middle genus is eventually also covered $k$ times.

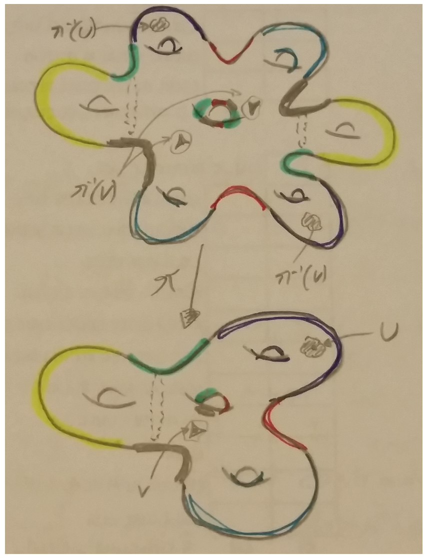


Figure 1: Courtesy of Gil Goffer. This picture represents the space $X$ with $n = 3$ (bottom) and its cover $\widetilde{X}$, with $k = 2$. We can see that each leaf was coloured to match the leaves in the cover that correspond to that in the original space. If the covering map is $\pi$, we can see the pre-images of the different neighbourhoods, $U$ and $V$.

# Problem 1

## Torus

We know the n-torus is $S^1 \times S^1 \times ... \times S^1$, thus its universal cover is easily identified as $\mathbb{R}^n$ and its fundamental group as $\mathbb{Z}^n$.

## Cylinder

The cylinder is written as $\mathbb{R} \times S^1$, hence its universal cover is $\mathbb{R}^2$, since $\mathbb{R}$ is its own cover as it is simply connected and it is also the universal cover of $S^1$. Again, since $\mathbb{R}$ is contractible, $\pi_1(\mathbb{R} \times S^1) = \pi_1(S^1) = \mathbb{Z}$.

## Projective Plane

We can think of the projective plane as the sphere under an equivalence relation that joins two points in the same line. Hence $X = RP^2 = S^2/x \sim -x$.

Thus by defining a map $P : S^2 \to S^2/x \sim -x$ by $x \mapsto \{x, -x\}$ we can see how $S^2$ covers $RP^2$ and since it is simply connected, it is its universal cover. Now in order to find the loop we must look at the group of deck transformations of $X$. Since the only non-trivial map that preserves fibers is exactly $f : S^2 \to S^2$ as $f(x) = -x$, and this transformation has order two as an element of the group, then we can say that $\pi_1(RP^2) = \mathbb{Z}/\{2\mathbb{Z}\}$.

## Bouquet of $n$ circles

As we saw in the targul, in a generalized way, the fundamental group of the bouquet of $n$ circles, glued at the same point, is the free group generated by $n$ elements. Hence $\pi_1(S^1 \vee ... \vee S^1) = F_{<a_1,...,a_n>}$. Furthermore, its universal cover is the Cayley graph of the free group, which in this case would have $n$ generators. It is contractible and covers all the bouquets. Another way of seeing it inductively, using a two dimensional representation of the graph, where one of the axis is $k < n$ dimensional.

## Finite Graph

For a finite graph, we will proceed in a similar way as for the bouquet of $n$ circles. First let us find a maximal spanning tree, $H < G$. This tree does not have any loops so it is contractible, hence $G$ is homotopically equivalent to its quotient by $H$. We can easily check that the quotient $G/H$ is homotopically equivalent to the bouquet of $m = |G| - |H|$ circles (an example represented in the figure below), since $m$ is exactly the number of edges that are not part of the tree and that will be glued to a single point, each generating one circle. Hence we conclude that $\pi_1(G) = \pi_1(G/H) = \pi_1(S^1 \vee ... \vee S^1) = F_{<a_1,...,a_m>}$, the free group of $m$ generators.

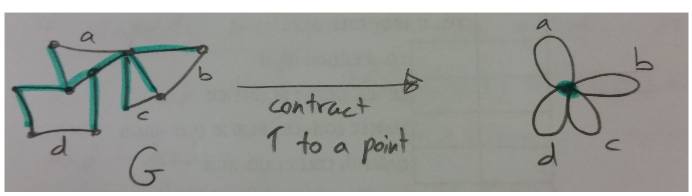


Figure 2: Courtesy of Gil Goffer. This picture depicts the homotopy between the maximal spanning tree and the bouquet of the spheres, glued at the same point.

## Problem 2

### (a)

First let us look at the bouquet $Y = S^2 \vee D^1 \vee S^2$. These are glued at a special point $(s_0, 0, s_0') \in (S^2, D^1, S^2)$. Since the disc is contractible to a point, this means that $Y$ is homotopically equivalent to $S^2 \vee S^2$. This equivalence is achieved when we make the equivalence relation $s_0 \sim D^1 \sim s_0'$ by identifying them with one new base point $\widetilde{s_0}$. It is easy to write this equivalence since we can easily show that $D^1 \simeq \{0\}$.

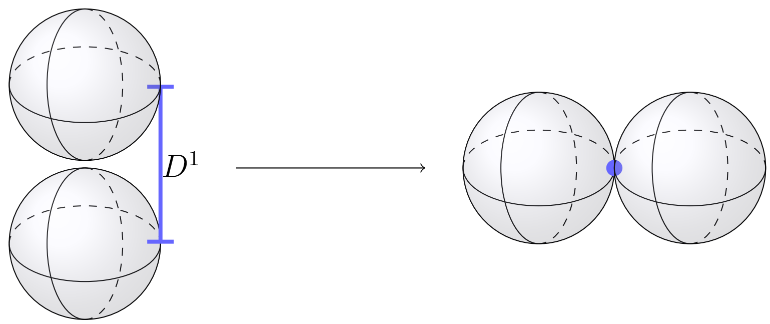


Figure 3: This figure shows the homotopic equivalence between a triple bouquet of a sphere, a cirlce and a sphere (with two 'different' base points) with a bouquet of two spheres, glued at a common base point.Note that here, $D^1$ is represented as a line for convenience of drawing, instead of drawing a segment of the helix.

This is easily generalized when we think of infinitely many of these triples (with a $D^1$ disc in between each of them) and infinitely many base points. The way to contract them is by thinking of this as having a sphere glued at each integer on the real line. We pick any of these, $S_0^2$, and we look simultaneously at the next one on each side and contract the intervals to the base point of $S_0^2$. Then we contract each consecutive $D^1$ at half the speed of the previous one, into the same base point. This will make each $D^1$ and the base points of each sphere to be equivalent under the identification relation. We do this as follows. Let $F : I \times Y \to X$ be a homotopy such that in the first half of the interval $I$, the first disc on each side of $S_0^2$ is contracted into the base point of $S_0^2$, thus joining the three spheres at a common base point. Then on the interval $[\frac{1}{2}, \frac{3}{4}]$ the next discs on each side are contracted, joining the two spheres to the previous three at a common base point. We continue in this manner, by halving the time of each contraction for every following disc (on both sides). In this way, since the series of timings converges to 1, we have that in finite time we contract $\mathbb{R}$ into a point, where at each integer there is a sphere glued.

Thus we have showed that the space $... \vee D^1 \vee S^2 \vee D^1 \vee ...$ is homotopically equivalent to the infinite bouquet of spheres glued at one point.

### $(b)$

If we let $Y = S^2 \vee \mathbb{R}$, we can see how this is the universal cover for $S^2 \vee S^1$ since $S^2$ is simply connected so it covers itself and $\mathbb{R}$ is the universal cover of $S^1$. Since they are glued at a point, this works.

## Problem 3

### $(a)$

Let $X$ and $Y$ be two spaces such that $\Omega(X)$ and $\Omega(Y)$ are the loops spaces respectively. Then we can define $\Omega(X) \times \Omega(Y)$ by the map $f : \Omega(X \times Y) \to \Omega(X) \times \Omega(Y) = \Omega$. Let $\omega \in \Omega(X \times Y)$, then $\omega : I \to X \times Y$ is a loop in two variables such that $\omega(0) = \omega(1) = (x_0, y_0)$ and $\omega(t) = (x_t, y_t)$, which means that every loop in the product space actually is divided into two coordinate-wise loops. Thinking of it like this, we let $f(\omega) = (\omega_X, \omega_Y)$ where each $\omega_X$ and $\omega_Y$ are the coordinate-wise loops so that $\omega_X(t) = x_t$ and $\omega_Y(t) = y_t$, with the same end points as the corresponding coordinates $x_0, y_0$ respectively. Clearly $f$ is invertible since every loop in the product space of loops can be put together to form a loop on a product of two spaces. Hence the two are equivalent.

Now we just need to show that the loops that are homotopic equivalent in $\Omega(X \times Y)$ induce homotopic equivalent loops coordinate-wise in $\Omega$, thus showing that the equivalent classes are the same in both and proving the first part. Let $\gamma \sim \omega$ in $\Omega(X \times Y)$, thus there is a homotopy $H : I \times I \to X \times Y$ such that $H(0, t) = \gamma$ and $H(1, t) = \omega$. Now, this induces two homotopies $H_X : I \times I \to X$ and $H_Y : I \times I \to Y$ such that $H_X(0, t) = \gamma_X$, $H_Y(0, t) = \gamma_Y$ and $H_X(1, t) = \omega_X$ and $H_Y(1, t) = \omega_Y$. Hence we get that $(\gamma_X, \gamma_Y) \sim (\omega_X, \omega_Y) \in \Omega$. Again, we can easily check that the backward direction holds too, hence showing that the homotopies in one space yield unique homotopies in the other space, thus there is a bijection between the equivalence classes and we conclude that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

$(b)$

Let $\widetilde{X}, \widetilde{Y}$ be the universal covers of $X$ and $Y$ respectively, with maps $p_X$ and $p_Y$. Then we consider $\widetilde{X} \times \widetilde{Y}$ as the universal cover of $X \times Y$ by the map $p : \widetilde{X} \times \widetilde{Y} \to X \times Y$ by $p(\widetilde{x}, \widetilde{y}) = (p_X(\widetilde{x}), p_Y(\widetilde{y}))$. This a cover map for $X \times Y$ since for every $(x, y) \in X \times Y$, there exists $x \in U \subseteq X$ and $y \in V \subseteq Y$ where $U$ and $V$ are open, such that $p_X^{-1}(U)$ and $p_Y^{-1}(V)$ are disjoint union of open neighbourhoods, mapped homeomorphically onto $U$ and $V$ respectively. Then take $U \times V$ an open neighbourhood of $(x, y) \in X \times Y$ such that $p^{-1}(U \times V) = p_X^{-1}(U) \times p_Y^{-1}(V)$, which is a disjoint union of open neighbourhoods homeomorphic to $U \times V$, so we have showed that $\widetilde{X} \times \widetilde{Y}$ is a cover for $X \times Y$, we just need to show that it is the universal cover.

In order to prove this final step, let us use part $(a)$ of this problem. Since we know that the equivalent classes are preserved under the product action of two spaces, since $\widetilde{X}$ and $\widetilde{Y}$ have trivial fundamental groups (as they are simply connected), then it follows that $\widetilde{X} \times \widetilde{Y}$ also has trivial fundamental group, hence it is simple connected and we have showed that $\widetilde{X} \times \widetilde{Y} = \widetilde{X \times Y}$. the universal cover of $X \times Y$.

# Problem 4

First of all, we know that $\mathbb{R}^2 \backslash \{n \text{ points}\}$ is homotopically equivalent to the wedge of $n$ circles, $S^1 \vee ... \vee S^1$, thus clearly their fundamental groups are isomorphic, i.e.: $\pi_1(\mathbb{R}^2 \backslash \{n \text{ points}\}) = F_{<a_1,...,a_n>}$ and similarly $\pi_1(\mathbb{R}^2 \backslash \{n - 1 \text{ points}\}) = F_{<b_1,...,b_{n-1}>}$. Hence we can just see how the inclusion occurs by identifying one of the generators $a_i$ with the identity $e$, which in this case is the class of constant loops. Thus we need to find $0 \neq x \in F_{<a_1,...,a_n>}$, which is some word in the free group and a loop over $\mathbb{R}^2 \backslash \{n \text{ points}\}$ that will map to the constant loop by $i_*(x)$. Now let us say that without loss of generality, we identify $a_1$ as the identity in the inclusion $i$. We will look at two examples and then prove the general case.

Let $n = 1$, we just let $x = a_1$.

Let $n = 2$, we take the word of the form $x = a_1a_2a_1^{-1}a_2^{-1}$ and thus by mapping $i_*(x) = i_*(a_1a_2a_1^{-1}a_2^{-1}) = ea_2ea_2^{-1} = a_2a_2^{-1} = e$.

Thus assume $n = k$, let $x = a_1a_ja_1^{-1}a_j^{-1}$ for any $j = 0, ..., k$, where we map $a_1$ by the inclusion to the identity. Hence, $i_*(x) = e$, as before. And we know that $x \neq e$ since $x$ is in the free group.

# Problem 5

Let $\widetilde{X}$ be the cover of $X$. In order to prove this, we will first look at a neighbourhood of any point in $y_1 \in Y$. Since $f_t(Y) = f(Y, t)$ is continuous, every point $(y_1, t) \in Y \times I$ has a product neighbourhood $N_t \times (a_t, b_t)$ such that $f(N_t \times (a_t, b_t)) \subset U_t \subset X$ for some neighbourhood $U_t$ of $f(y_1, t)$. By compactness of $\{y_1\} \times I$, finitely many products $N_t \times (a_t, b_t)$ cover $\{y_1\} \times I$. This implies that we can chose a single neighbourhood $y_1 \in N$ and a partition of $I$, such that for each cell of the partition, $f(N \times [t_i, t_{i+1}])$ is contained in some neighbourhood, call it $U_i$. Now we can start constructing the lift homotopy $\widetilde{f}_t(y) = \widetilde{f}(y, t)$. Assuming by induction that we have constructed the lift on $N \times [0, t_1]$, starting by the given $\widetilde{f}_0$ on $N$. Then we have that since $\widetilde{X}$ is a cover of $X$, there exists a set $\widetilde{U}_i \subset \widetilde{X}$ projecting homeomorphically onto $f(N \times [t_i.t_{i+1}]) \subset U_i$ by the covering map $p$, such that $\widetilde{f}(y_1, t_i) \in \widetilde{U}_i$. After replacing $N$ with a smaller neighbourhood of $y_1$, we may assume that $\widetilde{f}(N \times \{t_i\})$ is contained in $\widetilde{U}_i$, namely by replacing $N \times \{t_i\}$ by its intersection with $\widetilde{f}|_{N\times\{t_i\}}^{-1}(\widetilde{U}_i)$. Now we can define $\widetilde{f}$ on $N \times [t_i, t_{i+1}]$ to be the composition of $f_t$ with the homeomorphism $p^{-1} : U_i \to \widetilde{U}_i$. After a finite number of steps we eventually get a lift $\widetilde{f} : N \times I \to \widetilde{X}$ for some neighbourhood $N$ of $y_1 \in Y$. Clearly this a continuous map.

Finally, we need to make sure that this lift is unique for any point and neighbourhood we pick as we extend it to the whole of $Y$. Yet this follows simply by the path lifting property that we proved in class, where it states that it is a unique lift for one point in $Y$ (as this would make it a path). Thus, however, this implies that by continuity of $\widetilde{f}_t$ on a neighbourhood of a point in $Y$, we can say that it is continuous and unique on all of $Y$.

# EXTRA EXERCISES

## Problem 8

Let $p : C \to X$ be a covering map. Then let $A \subset C$ be open. Given $x \in p(A)$, chose a neighbourhood $U$ of $x$ that is evenly covered by $p$. Let $\{V_\alpha\}$ be the disjoint sets of the pre-image $p^{-1}(U)$. Since there exists a point $y \in A$ such that $p(y) = x$, then we must have that $y \in V_\beta$. Then we have that $V_\beta \cap A$ is open in $C$ and hence open in $V_\beta$. Now, because $p$ maps $V_\beta$ homeomorphically onto $U$, $p(V_\beta \cap A$ is open in $U$ and hence open in $X$, which is a neighbourhood of $x$ contained in $p(A)$, as desired.