ALGEBRAIC TOPOLOGY-EXC4

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Q0) Let X be path connected, and$\phi : \widetilde{X} \to X$ a covering map, show that:

(a) $\widetilde{X}$ is path connected iff $\pi_1(X)$ acts transitively on the fiber of x_0.

(b) $\left|\phi^{-1}(x_1)\right| = \left|\phi^{-1}(x_2)\right|$ for any $x_1, x_2 \in X$.

Proof. a) If $\widetilde{X}$is path connected then lets look at some fiber $\phi^{-1}(x_0)$. We take $y_1, y_2 \in \phi^{-1}(x_0)$ and connect them by a path $\widetilde{\gamma}$. Look at $\gamma = p \circ \widetilde{\gamma}$. This is a loop in X corresponds to $g = [\gamma]$. Notice that $g.y_1$ takes y_1 to the endpoint of the unique lift of γ that starting at y_1.But $\widetilde{\gamma}$ satisfy those conditions so it is the unique lift so $g.y_1 = \widetilde{\gamma}(1) = y_2$. Therefore the $\pi_1(X)$ acts transitively on the fiber of x_0.

b) We take a path γ from $x_1$to x_2. For each $x \in \gamma(I)$ take U_x such that $\phi^{-1}(U_x) \cong U_x \times D$. $\{U_x\}$ is a cover of $\gamma(I)$ and $\gamma(I)$ is compact so there exists a finite subcover $\{U_{x_i}\}_{i=1}^{n}$ of $\gamma(I)$. since I is connected, then also $\gamma(I)$ is connected. Lets assume that $d = \left|\phi^{-1}(x_1)\right| \neq \left|\phi^{-1}(x_2)\right| = k$. Lets look at $V_d$to be the union of all U_{x_i}such that $d = \left|\phi^{-1}(x_i)\right|$and $V_{nd}$to be the union of all U_{x_i}such that $d \neq \left|\phi^{-1}(x_i)\right|$. $(V_d \cap \gamma(I)) \cup (V_{nd} \cap \gamma(I)) = \gamma(I)$ and both are non empty so they must intersect- there exists a point $x \in V_d, V_{nd}$ so $x \in U_{x_i} \cap U_{x_j}$with $\phi^{-1}(x_i) \neq \phi^{-1}(x_j)$ and its impossible. □

Q1) a)Show that a covering map p is always open.

Proof. Let $V \subseteq \widetilde{X}$ then we need to prove that $p(V)$ is open. Let $x \in P(V)$ then it has a small neighborhood U_x such that $p^{-1}(U_x) \cong U_x \times D$. Then $p^{-1}(U_x) \cap V$ is open in $\widetilde{X}$ (from continuity of p). Now denote $\mu : p^{-1}(U_x) \longrightarrow U_x \times D$ the homeomorphism such that $proj \circ \mu = p$. Then:

$$p(p^{-1}(U_x) \cap V) = proj \circ \mu(p^{-1}(U_x) \cap V) = proj(\cup_{d \in D} (V_d \times \{d\})) = \cup_{d \in D} V_d \subseteq p(V)$$

And $\cup_{d \in D} V_d$ is open when $V_d = \mu(p^{-1}(U_x) \cap V)|_{d \in D}$ is open and it is open since W is open in $X \times D$ iff $W|_{X \times \{d\}}$is open for any $d \in D$. □

b)Let G act transitively on X, then $|G_x| = |G_y|$ for any $x, y \in X$.

Proof. Let G_x. and let $g \in G$ such that $g.x = y$. then $g^{-1}G_y g = G_x$. It is clear that $g^{-1}G_y g \subseteq G_x$ since for $k \in G_y$ we have $g^{-1}kg.x = x$. But also $gG_x g^{-1} \subseteq G_y$so $G_x \subseteq g^{-1}G_y g$ so we get $g^{-1}G_y g = G_x$ and left/right multiplication is a set theoretic isomporphism since there is an inverse. So $|G_x| = |G_y|$. □

Q2) Construct a non normal cover of $S^1 \vee S^1$:

Proof. We showed that a cover $\widetilde{X}$ is normal iff $\pi_1(\widetilde{X})$ is a normal subgroup. So the idea is to choose a non-normal subgroup H and take its corresponding cover. We choose $H =< a >$. Then it is not normal since $bab^{-1} \notin H$. Its cover looks like this:

We choose a point x_0 and attach a loop a to it, and another two "trees" T connected via 2 edges correponding to b and b^{-1}, when the trees T is the tree of the universal cover. See drawing attached (or Hatcher p.58 figure 12). It is clearly a cover in the two "trees" section and also at the other part as well and we can see that its fundemental group is $< a >$ since any entrance of a path γ to the "simply connected tree part" can be shrinked to the point x_0. □

Q3) Compute the fundamental group of the Klein bottle:

(a) Using its universal cover.

(b) Using Van-Kampen's theorem.

Proof. a) The Klein bottle can be described as $I \times I/(t, 0) \sim (1 - t, 1)$, $(0, s) \sim (1, s)$, so it can be cover by a torus by taking the 2 squares $I \times I$ and make a rectangle $[0, 1] \times [0, 2]$ when we glue them along the x axis. See figure. Now a torus can be covered by a plane so the plane also covers the Klein bottle.

Now we will find all the deck transformations on the universal cover. Notice that we have a grid described in the picture. The deck trasformation contains all the posible movements right (a) or left (a^{-1}) up by $(2b)$ or down $(2b)$ (and more). Notice that it is problematic to move up by (b) since then we glue the (a) edges in the wrong direction. I claim that the deck transformation group is generated by the translations $(m, n) \longmapsto (m + 1, n)$ and $(m, n) \longmapsto (-m, n + 1)$:

* It is trivial that the 2 types of translations are homeomorphism so we just need to prove that they preserves the fibers. The first translation clearly preserves the fibers since the covering space is symmetric according to this movement.

For the second translation, it is clearly preserves the fiber of the base point $(0, 0)$. Now, also notice that the edge a $\{(m, n), (m + 1, n)\}$ sent to the edge a $\{(-m, n + 1), (-m - 1, n + 1)\}$ so it preserves its orientation (sthe orientation changed when going up one point in the y axis) The same for b $\{(m, n), (m, n + 1)\}$ sent to the edge b $\{(-m, n + 1), (-m, n + 2)\}$ so the orientation also preserved. Also, from the way we constructed the cover (by first cover by a torus and then a lot of copies of the torus), it is clear that when looking at the square starting at (m,n) ($(m,n),(m,n+1),(m+1,n+1),(m+1,n)$) then it is a "flip" of the square above it $(m,n+1)$ and the square at $(m,n+1)$ is the same as $(-m,n+1)$ so by sending (m,n) to $(-m,n+1)$ we fix this flipping and get the "right" gluing. So $(m,n)\longmapsto(-m,n+1)$ is indeed a Deck transformation.

* We denote $\tau_1:(m,n)\longmapsto(m+1,n)$ and $\tau_2:(m,n)\longmapsto(-m,n+1)$. Notice that by those two translation we can generate all Deck transformation sending $(0,0)$ to (m,n). This follows from the fact that $\tau_2^2(m,n)=(m,n+2)$ so we if n is even then we can just compose τ_1,τ_2^2 the required times. If n is odd, then we send $(0,0)$ to $(-m,n-1)$ by applying τ_1,τ_2^2 the required times and then compose with τ_2 to send $(-m,n-1)$ to (m,n).

Since the deck transformation is determined by one point then, $<\tau_1,\tau_2>$is the group of deck transformations.

Notice that we can map $F_2=<a,b>$ to the Deck transformation group by taking $a\longmapsto\tau_1$ $b\longmapsto\tau_2$. it is obviously an homomorphism and surjective. Now, the kernel contains elements generated by $abab^{-1}$. In b) we will see that this is exactly the kernel and hence $<\tau_1,\tau_2>$ is exactly $<a,b>/abab^{-1}$

b) We can devide to Klein bottle into 2 Mobius strips with an intersection of $\cong S^1\times(0,1)$.See figure. By Van Kampen Theorem, $\phi:\pi_1(Mob_1)*\pi_1(Mob_2)\longrightarrow\pi_1(Klein)$ is surjective and the kernel is the normal subgroup generated by elements $i_{12}(w)i_{21}^{-1}(w)$ when $i_{12}:\pi_1(Mob_1\cap Mob_2)\longrightarrow\pi_1(Mob_1)$, $i_{21}:\pi_1(Mob_1\cap Mob_2)\longrightarrow\pi_1(Mob_2)$ and $w\in\pi_1(Mob_1\cap Mob_2)$. Now, Lets take the generator of $Mob_1\cap Mob_2$: it is maped by the inclutions to the loop on the boundery of the mobius and its corresponds to twice the generator of the mobius strip, i.e to x^2(See figure). So we have that:

$$N=<i_{12}(w)i_{21}^{-1}(w)>=<x^2\left(y^{-1}\right)^2>$$

Note that $<a,b>/abab^{-1}\cong<x,y>/x^2=y^2$ by the isomorphism $i(x)=b$ and $i(y)=ab$ notice that $i(y^2x^{-2})=abab^{-1}$ so it is well defined. Also we need to show that it is injective. The inverse map is $j(b)=x$ $j(a)=yx^{-1}$ so $j(abab^{-1})=y^2x^{-2}$ so it is well defined. Now $i\circ j(b)=b$ and $i\circ j(a)=i(yx^{-1})=abb^{-1}=a$ and the same $j\circ i(x)=x$ and $j\circ i(y)=j(ab)=y$ so i an isomorphism of groups. (its obvius that it is an homomorphism since its defined on the generators.) So We see that the deck transformation group can be represented as $<a,b>/abab^{-1}$. □

Q4) Prove a version of the Fundamental Theorem of Algebra, a non-constant polynomial $p(z)$ with coefficients in C has a root in C:

(a) Define $f_r(s)=\frac{p(r\cdot exp(2\pi is)}{|p(r\cdot exp(2\pi is))|})$, for a polynomial $p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_0$ with no roots in C. Show that $f_r(s)$ is a loop in $S^1\subseteq C$ for every r ≥ 0, and compute $[f_r]\in \pi_1(S^1)$. (b) Take $r>max(1,|a_0|+\cdots+|a_{n-1}|)$, and show that for $|z|=r$ and for every $t\in[0,1]$, we get that $p_t(z)=z^n+t(p(z)-z^n)$ has no roots on the circle $|z|=r$.

(c) Construct a homotopy between f_r and the loop $e^{2\pi ins}$, by recalling the non-triviality of $[exp(2\pi ins)]\in\pi_1(S^1)$ (why?), and the homotopy class of f_r, conclude that we must have that $n=0$ and thus that p(z) must be constant.

Proof. a) $f_r$is a loop in S^1 since we have $f_r(0)=f_r(1)=\frac{p(r)}{|p(r)|})=(1,0)$ when x is the real axis and y is the imaginary. Notice that we can define an homotopy $H(t,s)=f_{r(1-t)}(s)$ from $f_r(s)$ to the constant loop (we deine $f_0(s)=(1,0)$). This is well defined since $p(r\cdot exp(2\pi is)\neq 0$ for any r,s. H is continous since $H(t,s)=\frac{p(r(1-t)\cdot exp(2\pi is)}{|p(r(1-t)\cdot exp(2\pi is))|})$ which is a continous (as long as $p(z)\neq 0$ for any z). Notice that $H(t,0)=H(t,1)=(1,0)$ so this is a base preserving homotopy so $[f_r]$ is trivial.

b) We take $r>max(1,|a_0|+\cdots+|a_{n-1}|)$, then $|p_t(z)|=|z^n+t(p(z)-z^n)|\geq|z^n|-|t(p(z)-z^n)|$and for $|z|=r$ we have that $|z^n|=r^n$ and

$$|t(p(z)-z^n)|=\left|t(a_{n-1}z^{n-1}+...a_0)\right|\leq t\left|(|a_0|+\cdots+|a_{n-1}|)\,r^{n-1}\right|\leq tr^n$$

So $|p_t(z)|>0$ for $t\in(0,1)$ and for $t=0$ we have $|p_t(z)|=r^n$ and for $t=1$ we have that $|p_t(z)|=|p(z)|>0$ since $p(z)$and has no roots at all so in particulare on the circle above. So $p_t(z)$ has no roots on the circle $|z|=r$.

c) At first can creat an homotopy from $f_r$to $f_{r'}$such that $r'>max(1,|a_0|+\cdots+|a_{n-1}|)$ by $H_1(t,s)=f_{r+(r'-r)t}(s)$. Now we can creat the homotopy from $f_{r'}$ to $exp(2\pi ins)$ by:

$$H_2(s,t)=\frac{(r'\cdot exp(2\pi is))^n+t(p(r'\cdot exp(2\pi is))-(r'\cdot exp(2\pi is))^n)}{|(r'\cdot exp(2\pi is))^n+t(p(r'\cdot exp(2\pi is))-(r'\cdot exp(2\pi is))^n)|}$$

Again, since $r'>max(1,|a_0|+\cdots+|a_{n-1}|)$, from the argument in b) we have that H_2 is continous, and for $t=0$ we have $H_2(s,0)=(exp(2\pi is))^n=exp(2\pi ins)$ and for $t=1$ we have $H_2(s,1)=f_{r'}(s)$. so $f_{r'}(s)\sim exp(2\pi ins)$ and $exp(2\pi ins)$ make n loops counterclockwise, so $[exp(2\pi ins)]=n$. But $const\sim f_{r'}(s)\sim exp(2\pi ins)$, so $n=0$ and $p(z)$ constant. contradiction. □

Q5) Let $p:X_H\to X$ be a path connected covering space, where $G=\pi_1(X)$ and $H=\pi_1(X_H)$, and also set $G(X_H)$ to be the group of deck transformations, show that:

(a) $G(X_H)$ acts transitively on the fibers $\Longleftrightarrow$ the stabilizer of a point in the fiber under the action of $\pi_1(X)$ is normal.

(b) If H is normal then $G(X_H) \cong G/H$.

Proof. a) We showed in class that $G(X_H)$ acts transitively on the fibers iff $\pi_1(X_H)$ is normal in $\pi_1(X)$. But we also showed that $\pi_1(X)_{\widetilde{x_0}} = \pi_1(X_H)$ so it is normal. For the other direction- read backwards.

b) We showed in the "tirgul" that a Deck transformation is determined by its action on a single point. If H is normal then also $\pi_1(X)_{\widetilde{x_0}} = H$ is normal so $G(X_H)$ acts transitively on the fibers. Therefore, for any $\widetilde{x_0}, \widetilde{x_1} \in p^{-1}(x_0)$ there exists an isomorphism of covers $\varphi : X_H \longrightarrow X_H$ s.t $\varphi(\widetilde{x_0}) = \widetilde{x_1}$.

So we can define $\phi : \pi_1(X) \longrightarrow G(X_H)$ by $\phi([\gamma]) = \tau$ when τ is the deck transformation taking $\widetilde{\gamma}(0) = \widetilde{x_0}$ to $\widetilde{\gamma}(1) = \widetilde{x_1}$ when $\widetilde{\gamma}$ is the lift of γ. This map is well defined since $\pi_1(X)$ acts on a fiber $p^{-1}(x_0)$, and since $G(X_H)$ acts transitively on the fibers, then there always exists such deck transformation taking $\widetilde{\gamma}(0) = \widetilde{x_0}$ to $\widetilde{\gamma}(1) = \widetilde{x_1}$ and it is unique since Deck transformation is determined by its action on a single point.

ϕ is an homomorphism since $\phi([\gamma_1] * [\gamma_2])$ goes to the deck transformation takes $\widetilde{\gamma_1}(0) = \widetilde{x_0}$ to $\widetilde{\gamma_2}'(1) = \widetilde{x_2}$ when $\widetilde{\gamma_2'}$ is the lift starting from $\widetilde{\gamma_1}(1) = \widetilde{\gamma_2'}(0) = \widetilde{x_1}$. So $\phi([\gamma_1] * [\gamma_2]) = \tau = \tau_2 \circ \tau_1$ when $\tau_1$is the deck transformation from $\widetilde{x_0}$ to $\widetilde{x_1}$and τ_2 taking $\widetilde{x_1}$ to $\widetilde{x_2}$. Notice that $\phi([\gamma_1]) = \tau_1$ so we need to prove that $\phi([\gamma_2]) = \tau_1^{-1} \circ \tau_2 \circ \tau_1$ and the we will get that:

$$\phi([\gamma_1] * [\gamma_2]) = \tau_2 \circ \tau_1 = \tau_1 \circ \tau_1^{-1} \circ \tau_2 \circ \tau_1 = \phi([\gamma_1]) \circ \phi([\gamma_2])$$

But notice that if $\widetilde{\gamma_2}$ is a lift of γ_2 at $\widetilde{x_0}$ then $\tau_1\widetilde{(\gamma_2)}$ lifts γ_2 at $\widetilde{x_1}$ so by the uniqueness of the lifting (at a point) we get $\widetilde{\gamma_2'} = \tau_1\widetilde{(\gamma_2)}$ so $\widetilde{\gamma_2}(1) = \tau_1^{-1}\widetilde{\gamma_2'}(1) = \tau_1^{-1}(\widetilde{x_2})$ so $\phi([\gamma_2])$ takes $\widetilde{x_0}$ to $\tau_1^{-1}(\widetilde{x_2})$ and also $\tau_1^{-1} \circ \tau_2 \circ \tau_1$ takes $\widetilde{x_0}$ to $\tau_1^{-1}(\widetilde{x_2})$ so $\phi([\gamma_2]) = \tau_1^{-1} \circ \tau_2 \circ \tau_1$. So ϕ is an homomorphism.

The kernal of ϕ is the loops γ that lifts to loops that stabilize $\widetilde{x_0}$ so its $\pi_1(X)_{\widetilde{x_0}} = \pi_1(X_H) = H$ so we have an isomorphism $G/H \cong G(X_H)$. □

Q6) Construct a non-normal covering space of the Klein bottle by a Klein bottle and by a torus.

Proof. We construct a non normal cover by a Klein bottle. We make a 3-fold copy of the klein bottle as in the figure. This cover is a Klein bottle, so its fundemental group consists of all the loops generated by a and $b^3$with the relation ab^3ab^{-3}. Now, $p^*(Kcover) =< b^3, a > /abab^{-1}$. But it is not a normal subgroup of $< a, b > /abab^{-1}$ since if we take aba^{-1}it is not in $< b^3 > /abab^{-1}$. If $aba^{-1} \in< b^3, a > /abab^{-1}$ then also $b \in< b^3, a > /abab^{-1}$ and then $p^*(Kcover) = \pi_1(Klein)$ and then the cover is trivial. So This cover is not normal. □

Q7) Compute the fundamental group of the Hawaiian earring, that is the bouquet of countably many circles of radius $1/n$ and center $(1/n, 0)$ at their common point $(0, 0)$.

Proof. The fundemental group will be different from a countable bouquet of S^1, since in the countable bouquet of S^1, the image of all of the paths are inside a finite number of S^1(from compactness of I . But the hawaiian earring is a compact space and therefor we can do a path that cover infinite number of loops. So The fundemental group will be more close to a free group with infinite generators $*_{\alpha\in\mathbb{N}}\mathbb{Z}_\alpha$. But this group is too big since for example the path that takes the infinite concatenation of path on the first circle of radius 1 is not a valid loop (i.e continous) since this path need to the the loop in a faster and faster pace in order for the concatination to be infinite but then we loose the continuity. So we need to quatient out by those loops, so the fundemental group will be a quatient $*_{\alpha\in\mathbb{N}}\mathbb{Z}_\alpha/H$ where H consists of all the loops with "radius" that does not converge to 0 (and maybe some other conditions im not aware of). □

Q9) Show that generally, for any $H \leq G$ we have that $G(X_H) \cong N(H)/H$, where $N(H)$ is the normalizer of H.

Proof. See proof in Hatcher p.71 proposition 1.39. □