①

SECTION ALG TOP #1

* $X \cong Y$ are homeomorphic

$f: X \to Y$

$g: Y \to X$ $\quad g \circ f = id_X$

$\quad f \circ g = id_Y$

f, g cts. (f continuous, bijection, open map)

* $X \underset{\sim}{\simeq} Y$ are homot. equivalent

if $\quad f: X \to Y$ $\quad$ s.t. $\quad f \circ g \sim id_Y$

$\quad g: Y \to X$ $\quad g \circ f \sim id_X$

homotopic to id.

(There exists a homotopy) $H: X \times I \to X$

$H(x, 0) = g \circ f$

$H(x, 1) = id_X$

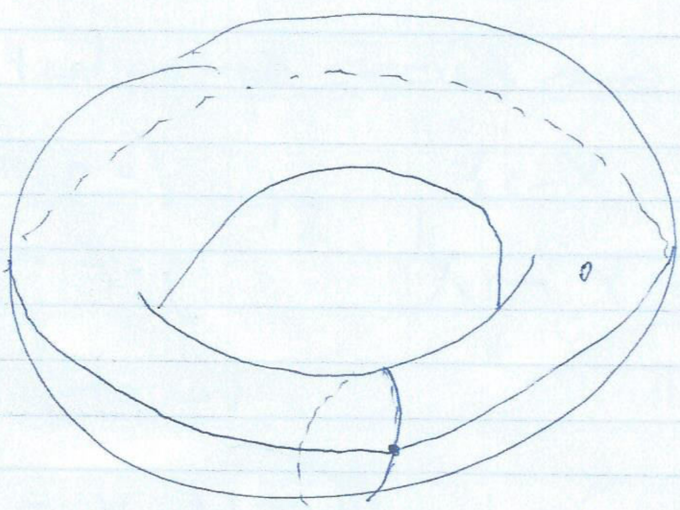
(and same for $f \circ g$ and id_Y)

<u>NOTE</u>: Homeomorphism $\Rightarrow$ homotopically equivalence.

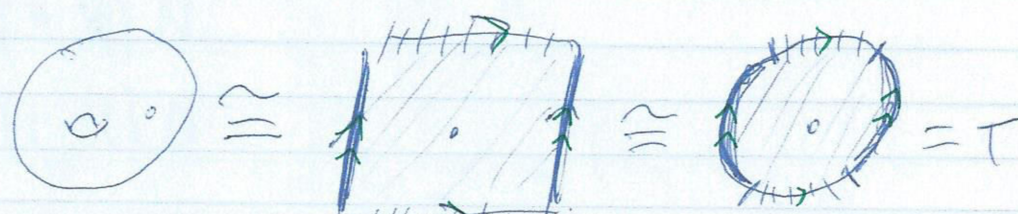
① Ex: show that torus w/o a point is hom-eq to a bouquet of 2 circles. $T\setminus\{*\} \simeq S^1 \vee S^1$

Bouquet/wedge: glueing at one point.

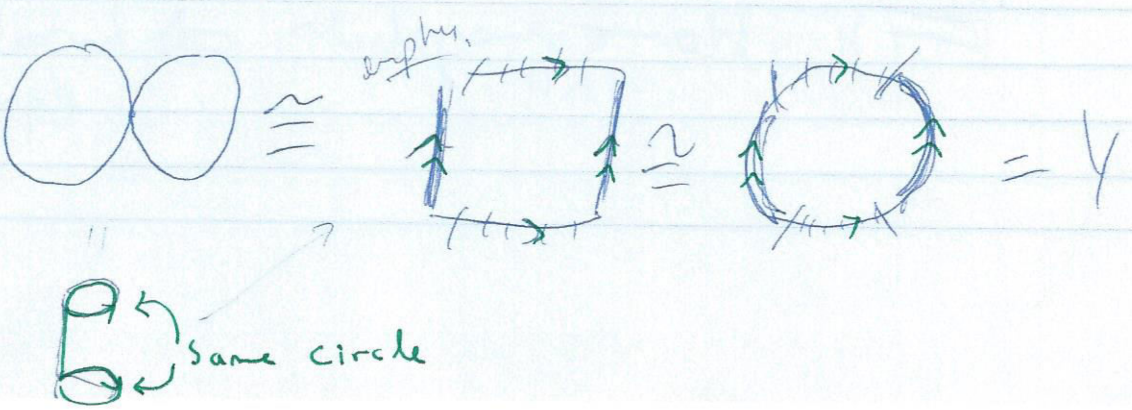
Show: $T^1 \setminus \{*\} \simeq S^1 \vee S^1$



trick



empty



③

$$f_1 : T \to Y$$
$$x \longmapsto \frac{x}{\|x\|}$$

$$f_2 : Y \longrightarrow T$$
$$x \longmapsto x$$

$$f_1 \circ f_2 = id_Y$$
$$x \longmapsto x \longmapsto \frac{x}{\|x\|} = x$$

$$f_2 \circ f_1 = \frac{x}{\|x\|} \overset{?}{\simeq} \underset{id_T}{x}$$

A homotopy between $f_2 \circ f_1$ and id_T.

(also $H(x,t)$ is continuous)

$$H(x,t) = \frac{x}{\|x\|} t + (1-t) x$$

$$\left.\begin{aligned} H(x,0) &= x \\ H(x,1) &= \frac{x}{\|x\|} \end{aligned}\right\} \Longrightarrow f_2 \circ f_1 \simeq id_T$$

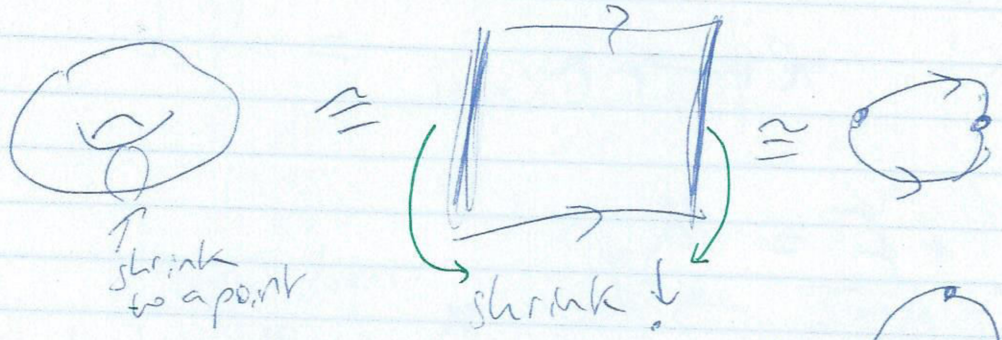
$$\Rightarrow \quad T \simeq Y.$$

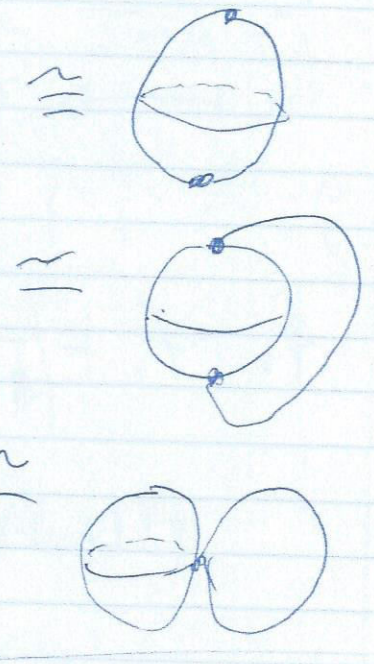
4

② **Ex:** Show torus mod a standard circle is homot. equiv to $S^1 \vee S^2$

$S^1 \times \{x_0\}$?

Solution:





Want: $X/A \simeq X$ if A is contractible

($id_A \simeq const_A$)

Def: (X, A) has homot. extension property (HEP) if

$g : X \times \{0\} \longrightarrow Y$

$f : A \times I \longrightarrow Y$

given f, g and (*), then can extend to a homotopy:

$H : X \times I \longrightarrow Y$

$$\boxed{f|_{A\times\{0\}} = g|_{A\times\{0\}}} \quad (*)$$

s.t. $H(a,t) = f(a,t)$

$H(x,0) = g(x)$

⑤

<u>Lemma:</u> if $A \times I \cup X \times \{0\}$ is ~~contractible~~ a retract of $X \times I$ then (X, A) has [HEP].

<u>Pf:</u> $\exists\, r : X \times I \longrightarrow A \times I \cup X \times \{0\}$

s.t. $r|_{A \times I \cup X \times \{0\}} = id$ ← (definition of retract)

then if given

$f : A \times I \longrightarrow Y$

$g : X \times \{0\} \longrightarrow Y$

$f|_{A \times \{0\}} = g|_{A \times \{0\}}$

need bigger function which is a homotopy and can restrict.

$K : A \times I \cup X \times \{0\} \longrightarrow Y$

$K|_{A \times I} = f$

$K|_{X \times \{0\}} = g$

$\Rightarrow H(x,t) = K \circ r(x,t)$

s.t. $H(x,0) = g(x)$
$H(a,t) = f(a,t)$ we extended f and g to H.

6

Lemma 2: If (X,A) has [HEP] and A is contractible then

$$X/A \simeq X$$ (Hatcher Prop 0.17)

Pf: Let $q : X \longrightarrow X/A$, $f_0 : A \to A$
the ~~[illegible]~~ quotient map

s.t. $f_0 = id_A \simeq const_A$
assumption

and f_t to be a homotopy extending f_0. (think of $f_t(x)$ as $H(x,t)$)

$$\left.\begin{array}{l} H : A \times I \to \overset{A}{\underset{\cup}{}} A \\ id_X : \underset{\underset{X \times \{0\}}{\shortparallel}}{X} \longrightarrow X \end{array}\right\} f_t$$

$f_t(A) \subseteq A$ ①, $\underbrace{q \circ f_t(A)}_{} = \{point\}$ ②

①+② $\Longrightarrow$ we can write the map $q \circ f_t$ as first q then something else factors through q.

$$q \circ f_t = \bar{f}_t \circ q \quad \text{where } \bar{f}_t : X/A \to X/A$$

Notice that $f_1(A) = \{\text{point}\}$

$$\Rightarrow f_1 = gq \quad \text{s.t. } g : X/A \longrightarrow X$$

will be the other side of the homotopy equivalence

But now, $q \circ g = \bar{f}_1$:

(so $\bar{x} \in X/A$)

$$q \circ g(\bar{x}) = qgq(x) = q f_1(x)$$
$$= \bar{f}_1 \circ q(x) = \bar{f}_1(\bar{x})$$

$$g : X/A \longrightarrow X \qquad \Rightarrow \quad q \circ g = \bar{f}_1(\bar{x}) \simeq \bar{f}_0(\bar{x}) = id_{X/A}$$

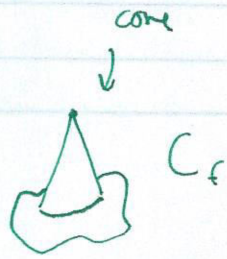
$$q : X \longrightarrow X/A \qquad g \circ q = f_1(x) \simeq f_0(x) = id_X$$

and we're done.

⊗ Show that if $f : X \to Y$ hom-equiv then mapping cone C_f is contractible

Pf: $X \times I \sqcup Y \Big/ \begin{smallmatrix} (x,0) \sim (x',0) \\ f(x) \sim (x,1) \end{smallmatrix}$ $\quad \forall x, x' \in X$

mapping cone of f



(8)

<u>Fact:</u> $f: X \to Y$ (for f which is a homotopy equivalence)

(*) $C_{f'} \simeq C_f$

$f': X \to f(x)$

Want to show: $id_{C_{f'}} \simeq const_{C_{f'}}$

$f \circ g \simeq id_X$

$g \circ f \simeq id_Y$

by assumption $\begin{cases} H_1(x,1) = id_X \\ H_1(x,0) = g \circ f(x) \end{cases}$

Define a homotopy $H((x,s),t) = (H_1(x,t), st)$ $\quad H: C_{f'} \times I \to C_{f'}$

$H((x,s),1) = (H_1(x,1), s) = (x,s) = id_{C_{f'}}$

$H((x,s),0) = (H_1(x,0), 0) =$ point const map in $X \times \{0\}$.

$(x,0) \sim (x',0) \quad \forall x, x' \in X$.

<u>(*) Proof of fact:</u> Assume f a hom. equivalence.

Define $i: X \times I \amalg f(X)/_\sim \to X \times I \amalg Y/_\sim$ by the inclusion $((x,t) \mapsto (x,t))$

$h: X \times I \amalg Y/_\sim \to X \times I \amalg f(X)/_\sim$ by $h(x,s) = (g \circ f(x), s)$

Note that h is well defined. $\quad h(y) = f \circ g(y) \sim (g(y), 1)$

Since $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$, $\quad \begin{array}{l} h \circ i \simeq id_{C_{f'}} \\ i \circ h \simeq id_{C_f} \end{array} \Rightarrow C_f \simeq C_{f'}$

$(x,s) \sim y \Leftrightarrow y = f(x),\ s = 1$

$h(x,1) = (g \circ f(x), 1) = (g(y), 1) = (f \circ g(y), 1) = h(y)$