Algebraic Topology - Exercise 10

Solve the following questions:

1. Let $f : X \to Y$ be a continuous map, and let $\varphi : Y \to C_f$ be the inclusion of Y into the mapping cone of f. Show that there are maps $f_1 : C_\varphi \to S(X)$ and $f_2 : S(X) \to C_\varphi$ such that f_1 restricted to C_X (the cone of X) maps onto $S(X) \backslash \{X \times \{1\}\}$ and that f_2 restricted to $\{(x,t) \in S(X) | t > \frac{1}{2}\}$ is the identity map of $\{(x,t) \in C_X | t \neq 1\}$.
   [**Note, this shouldn't be hard, and is mainly for you to convince yourself better of the Barret-Puppe construction.**]

2. Let $X = \bigcup_{i=0}^{\infty} X_i$ be a topological space with $X_0 \subseteq X_1 \subseteq \ldots$, X_i closed sets s.t. $C_{\eta_{n-1}} \simeq \bigvee S^n$, where $\eta_{n-1} : X_{n-1} \to X_n$ is the inclusion, and X_0 is discrete. Let h be a homology theory.
   (a) Show that there is a map $\tilde{\partial}_n : h_n(X_n/X_{n-1}) \to h_{n-1}(X_{n-1}/X_{n-2})$ (We will prove it in Exercise Sesion 11, no need to hand in.).
   (b) Prove that the maps $\tilde{\partial}_n$ together with h_n form a chain complex (that is, that $\tilde{\partial}_n \circ \tilde{\partial}_{n+1} = 0$).
   (c) Prove that for this complex we have that $h_n(X_{n+1}) = h_n(X_{n+k})$.
   (d) Deduce that if $X = \bigcup_{i=0}^{n} X_i$, then the homologies of the chain complex above are exactly $h(X)$.