Algebraic Topology Lecture 1

Notes by Daniel Benarroch

4/11/2014

1 Motivation

We want to find invariant properties of topological spaces so that when mapped to other topological spaces by "special maps" (i.e. homeomorphisms or homotopy equivalences), these properties are preserved.

For example, let X be a topological space, the following properties are some invariants of X:

Compactness A subset $W \subseteq X$ is a compact set if every open cover of W has a finite subcover.

Connectedness A subset $W \subseteq X$ is said to be connected if we can not find two disjoint open sets such that W is their union.

Path Connectedness A subset $W \subseteq X$ is said to be path connected if any two points in W are connected by a continuous path.

Note: Homeomorphisms preserve more invariants than homotopy equivalences, which are thus more "general" maps between topological spaces.

2 First Definitions

Definition 2.1 (Homotopy)**.** *Let* $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ *be continuous maps. We say that* $f \sim g$ *are* homotopic *if there exists a continuous* $\varphi : X \times I \to Y$ *such that* $\varphi|_{X\times\{0\}} = f$ *and* $\varphi|_{X\times\{1\}} = g$.

Definition 2.2 (Homotopy Equivalence)**.** *Two topological spaces X, Y are* homotopy equivalent *if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim Id$ and $g \circ f \sim Id$, that is homotopic to the identity map, we then write $X \simeq Y$.*

Remark 2.3. If $f \circ g$ and $g \circ f$ are not only homotopic to the identity map, but are actually equal to their respective identity maps, then f and g are actually homeomorphisms. The notion of homotopy equivalence therefore generalizes the notion of homeomorphism, as noted previously. In particular, all homeomorphic spaces are homotopy equivalent.

Example 2.4. Let $I = [0, 1]$, take $f : \{p\} \to I$ for a space of one point $\{p\}$, and $f(p) = 0 \in I$. Also, let $g : I \to p$, where $g(x) = p$ for all $x \in I$. Clearly, $g \circ f = Id_{\{p\}}$ and $f \circ g \sim Id_I$ by letting $F(x, t) = tx$, so at $t = 0$, we have that $f \circ g(x) = 0 \forall x \in I$ and at $t = 1$, that $f \circ g(x) = x \ \forall x \in I$.

3 Some Category Theory

Definition 3.1. *In the language of Category Theory, we will talk about the* category of topological spaces *where the objects in this category are topological spaces and for every two such spaces X and Y, we define the morphisms between them,* $\mathbf{Cont}(X, Y)$*, to be the set of all continuous functions between X and Y.*

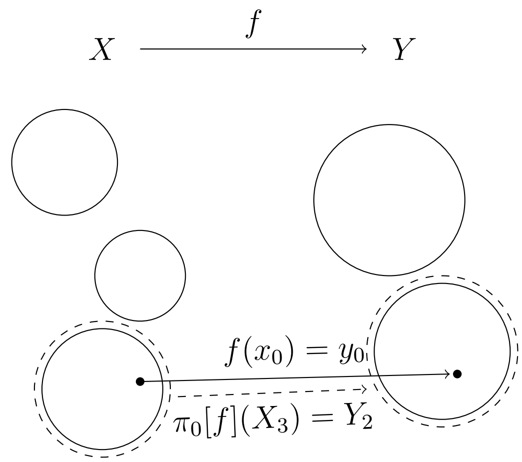
If we let $\sim$ to be the homotopic equivalence relation between continuous maps, we denote by $\mathbf{H}(X, Y) = \mathbf{Cont}(X, Y)/\sim$ the set of equivalence classes under homotopy, which is the set of maps between objects (which are topological spaces) in the *homotopical category.*

4 Invariant Sets

Let X be a topological space and let $I(X)$ be a set associated to X via some invariant, for example $\pi_0(X) = \{\text{path-connected components of } X\}$. We are interested in knowing when the invariant sets are equal up to set isomorphism, i.e bijection.

Let $f : X \to Y$ be a continuous map, we define $I_f : I(X) \to I(Y)$. This is a well-defined map of sets, as illustrated in the next example.

Example 4.1. Let $I(X) = \pi_0(X) = \{X_j\}_{j \in J}$, where X_j are the path components of X. We then write $I_f = \pi_0(f)$, where $f : X \to Y$ and $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$. Now, pick a point $x_0 \in X_j \in \pi_0(X)$, we require that $I_f(X_j) = Y_j$, where $f(x_0) = y_0 \in Y_j$. This makes I_f well defined.



Corollary 4.2. *If X and Y are homotopically equivalent, as in definition 2.2, then $I(X) \cong I(Y)$, as an isomorphism of sets, i.e., a bijection. For example 4.1, we get a bijection between the path-connected components, $\pi_0(X) \cong \pi_0(Y)$.*

Proof. Exercise number 5 in the first Exercise sheet. $\square$

5 Operations on Topological Spaces

Definition 5.1 (Pointed Topological Space)**.** *We say a topological space (X, x_0) is* pointed *when x_0 is given as a special point in X and for any other pointed topological space (Y, y_0) and a continuous map $f : X \to Y$, we have that $f(x_0) = y_0$.*

Definition 5.2 (Contractible)**.** *We say a pointed topological space (X, x_0) is* contractible *if (X, x_0) is homotopy equivalent to a one-point space, that is, if the identity map on X is homotopic to a constant map. (if we can write $f : X \times [0, 1] \to X$ such that $f(x, 0) = x$ and $f(x, 1) = x_0$.)*

Definition 5.3 (Cartesian Product)**.** *A* product *of topological spaces X and Y, $M = X \times Y$, is given by the following diagram,*

$$
\begin{tikzcd}
 & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y \\
 & S \arrow[ul, "\forall P_X"] \arrow[ur, "\forall P_Y"'] \arrow[uu, dashed, "\exists f"] &
\end{tikzcd}
.
$$

Where in this diagram, $f(s) = (P_X(s), P_Y(s))$ for all topological spaces S that have such maps. A subset $W \subseteq X \times Y$ is open if it is the product of a pair of open subsets of X and Y, respectively, or if it is the union of an arbitrary number of such subsets.

Definition 5.4 (Co-product)**.** *The* co-product *of two topological spaces X and Y is the dual object to the Cartesian product and is denoted by $X \sqcup Y$, which is the disconnected union of X and Y and is defined by the following diagram,*

$$
\begin{tikzcd}
 & S & \\
X \arrow[ur, "g_X"] \arrow[dr, hook, "i_X"'] & & Y \arrow[ul, "g_Y"'] \arrow[dl, hook', "i_Y"] \\
 & X \sqcup Y \arrow[uu, dashed, "\exists f"] &
\end{tikzcd}
.
$$

We define the topology on the co-product to be as follows. A subset $W \subseteq X \sqcup Y$ is open if $W \cap X$ and $W \cap Y$ are open in the spaces X and Y respectively.

In order to define the *quotient*, we need an equivalence relation E on $X \times X$ and we let X/E be the set of equivalence classes. We choose the weakest topology on X/E such that the quotient map is continuous.

Definition 5.5 (Quotient Map)**.** *Let $\mathfrak{q} : X \to X/E$ be the natural map sending each $x \in X$ to its equivalence class in X/E. This map is called the* quotient map*, and a subset of X/E is declared to be open if its pre-image in X is open.*

Example 5.6 (Quotient for pointed topological spaces)**.** Let $(Y, x_0) \subset (X, x_0)$ be a pointed subspace of a pointed topological space (with the same special point). For the sake of simplicity, write $X = (X, x_0)$ and $Y = (Y, x_0)$. Let the equivalence relation $E := Y \times Y \cup \Delta X \subset X \times X$, we define for pointed topological the quotient as $X/Y = X/E$.

Definition 5.7 (Bouquet)**.** *The* bouquet *(or wedge) is the co-product of pointed topological spaces. We write $X \vee Y := X \sqcup Y/\{x_0, y_0\}$. We can interpret this operation geometrically as gluing the two disjoint spaces, X and Y, at their respective* *special points x_0 and y_0. The following diagram helps us understand the co-product for pointed spaces.*

$$\begin{tikzcd}
 & (S, s_0) & \\
(X, x_0) \arrow[ur, "g_X"] \arrow[dr, "i_X"'] & & (Y, y_0) \arrow[ul, "g_Y"'] \arrow[dl, "i_Y"] \\
 & (X \vee Y, \{x_0, y_0\}) \arrow[uu, dashed, "\exists f"] &
\end{tikzcd}$$

Where each of the maps sends the special point to the corresponding special point. In the case of the inclusions, $i_X(x_o) = x_0$ and $i_Y(y_0) = y_0$.

Definition 5.8 (Smash Product)**.** *The* smash product *of two topological spaces is defined as $X \wedge Y = X \times Y / X \vee Y = X \times Y / (X \sqcap Y / \{x_0, y_0\})$. In simple terms we are looking at the equivalence classes of the bouquet relation over the product space of X and Y.*

Example 5.9. The smash product of the unit circle with the unit circle is a sphere, that is $S^1 \wedge S^1 = S^2$. If we consider the product $S^1 \times S^1$ as a square with equivalent parallel sides, we can easily look at the bouquet equivalence relation as making the square become a four petal rose, which specific arcs equivalent to others, as induced from the square we started with. This show $S^1 \times S^1$ is homeomorphic to a sphere.

Definition 5.10 (Cone, Suspension and Mapping Cylinder of X)**.**

Cone

$Cone(X) := X \times I / (X \times \{1\})$, meaning that if we span our space X over the interval I and we collapse all points $(x, 1)$ to a single point, as a representative of the equivalence class, we get the construction, which "looks like a cone". Note that cone(X) is contractible to a point.

Suspension

$Suspension(X) = S(X) = Cone(X)/(X \times \{0\})$, so following the cone construction, we now do the same kind of contraction to a point at the other end of the interval in order to get the suspension. As a small example, $S(S^1)$ looks like a two way cone, with S^1 as the base of both cones, which is homeomorphic to a sphere.

Reduced Suspension for pointed topological spaces
$\Sigma(X, x_0) = \Sigma X = S(X)/(x_0 \times I)$ where X means (X, x_0) as a pointed topological space. The reduced suspension only exists for pointed topological spaces. As an exercise we can show that $\Sigma X = S^1 \wedge X$.

Cylinder and Mapping Cylinder The cylinder of X, $Cyl(X) = C(X)$, *is literally what is sounds like, making our space X into a cylinder by spanning it over an interval I. Now, if $f : X \to Y$, we call the* cylinder of f, $Cyl(f) = C_f = (C(X) \sqcup Y)/((x_0, 0) \sim f(x))$, *where we are gluing X to Y through the image of X in Y. Finally, $Cone(f) := C_f/(X \times \{1\})$ is called the* mapping cone *of f. We can think of it as going one step further and glue all the points on the other end of $C(X)$ at $t = 1$. This construction is known as the "witch hat".*

Remark 5.11. We have an alternative way of constructing the suspension of X. Let $i : X \to Cone(X)$ be the inclusion of $X \subseteq Cone(X)$ by identifying X with $(X, 1)$. Then we let $suspension(X) = S(X) = C_i$, the mapping cylinder of i.

Definition 5.12 (Push-out)**.** *We define the* push-out *of two spaces as a different form of gluing. Instead of gluing two disjoint spaces X and Y at one point, we glue them along a subset Z of X and Y and we say that we glue them through Z. Let $Z \subseteq X$ and $Z \subseteq Y$ we write $G_Z = X \sqcup_Z Y = X \sqcup Y/\{f(z) \sim g(z)\}$, where $f : Z \hookrightarrow X$ and $g : Z \hookrightarrow Y$ as shown in the diagram below.*

$$\begin{tikzcd}
& X \sqcup_Z Y & \\
X \arrow[ur, "i_X"] & & Y \arrow[ul, "i_Y"'] \\
& Z \arrow[ul, hook, "f"] \arrow[ur, hook, "g"'] &
\end{tikzcd}$$

It is clear how we can glue these two sets together through Z.