Generalized Functions Exercise 5

Shai Keidar

1.

$$\begin{aligned} C^{-\infty}(V)\otimes E &= C_c^\infty(V,\mathrm{Haar}(V))^*\otimes E \\ &\cong (C_c^\infty(V)\otimes \mathrm{Haar}(V))^*\otimes E \\ &\cong (C_c^\infty(V)\otimes \mathrm{Haar}(V)\otimes E^*)^* \\ &= (C_c^\infty(V)\otimes (\mathrm{Haar}(V)\otimes E^*))^* \\ &\cong C_c^\infty(V,\mathrm{Haar}(V)\otimes E^*)^* \end{aligned}$$

2. In order to define an embedding $C_c^\infty(V,E)\hookrightarrow C^{-\infty}(V,E)$ it is enough to find an embedding $C_c^\infty(V)\hookrightarrow C^{-\infty}(V)$ since $C_c^\infty(V,E)=C_c^\infty(V)\otimes E$ and $C^{-\infty}(V,E)=C^{-\infty}(V)\otimes E$.

   Let $f\in C_c^\infty(V)$. Define $\xi_f : C_c^\infty(V)\times \mathrm{Haar}(V)\to\mathbb{R}$ by

$$\xi_f(g,\mu) := \int_V fg\,d\mu$$

   Note that ξ_f is bilinear:

$$\begin{aligned} \xi_f(a_1g_1+a_2g_2,\mu) &= \textstyle\int_V f(a_1g_1+a_2g_2)\,d\mu \\ &= a_1\textstyle\int_V fg_1\,d\mu + a_2\int_V fg_2\,d\mu \\ &= a_1\xi_f(g_1,\mu)+a_2\xi_f(g_2,\mu) \end{aligned}$$

$$\begin{aligned} \xi_f(g,a_1\mu1+a_2\mu_2) &= \textstyle\int_V fg\,d(a_1\mu_1+a_2\mu_2) \\ &= a_1\textstyle\int_V fg\,d\mu_1 + a_2\int_V fg\,d\mu_2 \\ &= a_1\xi_f(g,\mu_1)+a_2\xi_f(g,\mu_2) \end{aligned}$$

   Whence it defines a linear function

$$\xi_f : C_c^\infty(V,\mathrm{Haar}(V)) = C_c^\infty(V)\otimes\mathrm{Haar}(V)\to\mathbb{R}$$

i.e. $\xi_f \in C^{-\infty}(V) = C_c^\infty(V, \mathrm{Haar}(V))^*$. We got a function $\xi : C_c^\infty(V) \to C^{-\infty}(V)$ given by

$$\langle \xi_f, g \otimes \mu \rangle = \int_V fg \, d\mu$$

It is obviously linear. Assume that $\xi_f = 0$ for some f, i.e.

$$\int_V fg \, d\mu = 0 \;\forall g \in C_c^\infty(V), \mu \in \mathrm{Haar}(V)$$

Fix some non-zero Haar measure μ. Assume that $f(x) \neq 0$ for some x. Wlog $f(x) > 0$. Let U be a neighborhood of x s.t. $f > 0$ on U. Choose a bump function g s.t. $g > 0$ on U and $g = 0$ outside of U. Then

$$0 = \int_V fg \, d\mu = \int_U fg \, d\mu$$

But since $fg > 0$ on U, $\mu(U) = 0$ in contradiction, as μ is a Haar measure.

3. Let V be an n-dimensional vector space.

$$\Omega^{\mathrm{top}}(V) = \Omega^n(V) = \Lambda^n(V^*) \cong \{f : V^n \to \mathbb{R} \text{ multilinear and anti-symmetric}\}$$

We show that $f : V^n \to \mathbb{R}$ is multilinear and anti-symmetric if and only if $f(Av_1, \ldots, Av_n) = \det(A) f(v_1, \ldots, v_n)$ for any $A \in \mathrm{End}(V)$:
Let $f : V^n \to \mathbb{R}$ be a multilinear and anti-symmetric function. Choose a basis $e_1, \ldots, e_n$ for V, and write $A = (a_{i,j})$ as a matrix w.r.t it. First notice that

$$\begin{aligned}
f(Ae_1, \ldots, Ae_n) &= f(\textstyle\sum_j a_{1,j} e_j, \ldots, \sum_j a_{n,j} e_j) \\
&= \textstyle\sum_{j_1,\ldots,j_n} a_{1,j_1} \cdots a_{n,j_n} f(e_{j_1}, \ldots, e_{j_n}) \\
&\qquad \text{Since } f \text{ is multilinear} \\
&= \textstyle\sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{n,\pi(n)} f(e_{\pi(1)}, \ldots, e_{\pi(n)}) \\
&\qquad \text{Since } f(e_{j_1}, \ldots, e_{j_n}) = 0 \text{ if } j_i = j_{i'} \text{ for some } i, i' \\
&= \textstyle\sum_{\pi \in S_n} \mathrm{sign}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)} f(e_1, \ldots, e_n) \\
&\qquad \text{Since } f \text{ is anti-symmetric} \\
&= \det(A) f(e_1, \ldots, e_n)
\end{aligned}$$

Now let $v_1, \dots, v_n \in V$. Write $B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$.

$$\begin{aligned} f(Av_1, \dots, Av_n) &= f(ABe_1, \dots, ABe_n) \\ &= \det(AB) f(e_1, \dots, e_n) \\ &= \det(A)\det(B) f(e_1, \dots, e_n) \\ &= \det(A) f(Be_1, \dots, Be_n) \\ &= \det(A) f(v_1, \dots, v_n) \end{aligned}$$

Now, assume that $f : V^n \to \mathbb{R}$ satisfies $f(Av_1, \dots, Av_n) = \det(A) f(v_1, \dots, v_n)$ for any $A \in \mathrm{Aut}(V)$. Choose a basis $e_1, \dots, e_n$ for V, and write $C := f(e_1, \dots, e_n)$. Let $v_1, \dots, v_n$, and write $B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. Thus

$$f(v_1, \dots, v_n) = f(Be_1, \dots, Be_n) = \det(B) f(e_1, \dots, e_n) = C \det(v_1, \dots, v_n)$$

So $f = C \cdot \det$ and thus is multilinear and anti-symmetric.

4. (a) We define a function $\phi : \mathrm{Haar}(W) \times \mathrm{Haar}(V/W) \to \mathrm{Haar}(V)$ in the following way: Let $(\mu, \nu) \in \mathrm{Haar}(W) \times \mathrm{Haar}(V/W)$ and let $f \in C_c(V)$. For every $\alpha = v + W \in V/W$ define $f_\alpha = \int_W f(v+w)\, d\mu$. Since μ is translation-invariant, it does not depend on the choice of representative. Define

$$\langle f, \phi(\mu, \nu) \rangle := \int_{V/W} f_\alpha \, d\nu(\alpha) = \int_{V/W} \int_W f(v + w)\, d\mu(w)\, d\nu(v + W)$$

ϕ is well defined, i.e. $\phi(\mu, \nu)$ is a Haar measure: Let $x \in V$, and look at $f_x(v) := f(x + v)$.

$$\begin{aligned} \langle f_x, \phi(\mu, \nu) \rangle &= \textstyle\int_{V/W} \int_W f_x(v + w)\, d\mu(w)\, d\nu(v + W) \\ &= \textstyle\int_{V/W} \int_W f(x + v + w)\, d\mu(w)\, d\nu(v + W) \\ &= \textstyle\int_{V/W} \int_W f(x + v + w)\, d\mu(w)\, d\nu(x + v + W) \text{ (ν is a Haar measure)} \\ &= \textstyle\int_{V/W} \int_W f(v + w)\, d\mu(w)\, d\nu(v + W) \\ &= \langle f, \phi(\mu, \nu) \rangle \end{aligned}$$

It is also easy to see that ϕ is bilinear, since it is bilinear w.r.t scalar multiplication and Haar(W) and Haar(V/W) are both one-dimensional. So ϕ defines a morphism $\overline{\phi}$: Haar(W)$\otimes$Haar(V/W) $\to$ Haar(V) by

$$\langle f, \overline{\phi}(\mu \otimes \nu)\rangle = \int_{V/W} \int_W f(v+w)\, d\mu(w)\, d\nu(v+W)$$

Since $\overline{\phi}$ is not 0 and both spaces are one-dimensional, it is an isomorphism.

(b) Let $B_1 = \{w_1, \ldots, w_p\}$ be a basis of W and $B_2 = \{v_1 + W, \ldots, v_q + W\}$ be a basis of V/W, so $B = \{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ is a basis of V. We know that using those bases, the spaces Ω^{top} equal Span{det}, so we define $\Omega^{\text{top}}(W) \otimes \Omega^{\text{top}}(V/W) \to \Omega^{\text{top}}(V)$ by

$$(a \det_{B_1}) \otimes (b \det_{B_2}) \mapsto ab \det_B$$

It is obviously an isomorphism of linear spaces. Note that it does not depend on the choices we made: Let $B_1' = \{w_1', \ldots, w_p'\}$ be another basis for W and $B_2' = \{v_1' + W, \ldots, v_q' + W\}$ be a basis for V/W (or same basis with other representatives). We let $B' = \{w_1', \ldots, w_p', v_1', \ldots, v_q'\}$ be a basis for V. Now

$$\begin{aligned} \det_{B_1'} &= \det(M_{B_1}^{B_1'}) \det_{B_1} \\ \det_{B_2'} &= \det(M_{B_2}^{B_2'}) \det_{B_2} \\ \det_{B'} &= \det(M_B^{B'}) \det_B \end{aligned}$$

Notice that

$$M_B^{B'} = \begin{pmatrix} M_{B_1}^{B_1'} & 0 \\ * & M_{B_2}^{B_2'} \end{pmatrix}$$

So $\det(M_B^{B'}) = \det(M_{B_1}^{B_1'}) \det(M_{B_2}^{B_2'})$ and therefore, using both choices, $\det_{B_1'} \otimes \det_{B_2'} = \det(M_{B_1}^{B_1'}) \det_{B_1} \otimes \det(M_{B_2}^{B_2'}) \det_{B_2}$ would map to $\det_{B'} = \det(M_B^{B'}) \det_B = \det(M_{B_1}^{B_1'}) \det(M_{B_2}^{B_2'}) \det_B$, so the isomorphism does not depend on our choices.

(c)

$$\begin{aligned}\mathrm{Ori}(V) &= \Omega^{\mathrm{top}}(V) \otimes |\Omega^{\mathrm{top}}(V)| \\ &= \Omega^{\mathrm{top}}(W) \otimes \Omega^{\mathrm{top}}(V/W) \otimes |\Omega^{\mathrm{top}}(W) \otimes \Omega^{\mathrm{top}}(V/W)| \\ &= \Omega^{\mathrm{top}}(W) \otimes \Omega^{\mathrm{top}}(V/W) \otimes |\Omega^{\mathrm{top}}(W)| \otimes |\Omega^{\mathrm{top}}(V/W)| \\ &= \Omega^{\mathrm{top}}(W) \otimes |\Omega^{\mathrm{top}}(W)| \otimes \Omega^{\mathrm{top}}(V/W) \otimes |\Omega^{\mathrm{top}}(V/W)| \\ &= \mathrm{Ori}(W) \otimes \mathrm{Ori}(V/W)\end{aligned}$$

(d) Remember that over non-archemedian fields, $S(X) = C_c^\infty(X)$. We saw that over non-archemedian fields, since $W \subseteq V$ is closed, we have an exact sequence

$$0 \to S(V \setminus W) \to S(V) \to S(W) \to 0$$

And thus, since $\mathrm{Dist}(X) = S^*(X)$, we have an exact sequence

$$0 \to \mathrm{Dist}(W) \to \mathrm{Dist}(V) \to \mathrm{Dist}(V \setminus W) \to 0$$

So we have an inclusion $\mathrm{Dist}(W) \hookrightarrow \mathrm{Dist}(V)$ with image equal to $\mathrm{Ker}(\mathrm{Dist}(V) \to \mathrm{Dist}(V \setminus W)) = \{\xi \in \mathrm{Dist}(V) \,|\, \mathrm{supp}(\xi) \subseteq W\} = \mathrm{Dist}_W(V)$. Therefore $\mathrm{Dist}(W) \cong \mathrm{Dist}_W(V)$.

(e) Let $B = \{e_1, \ldots, e_n\}$ be a basis of V and $C = \{f_1, \ldots, f_n\}$ it's dual basis. Let M and P be the parallelepipeds spanned by B and C respectively. Let $\mu \in \mathrm{Haar}(V)$ be the Haar measure satisfying $\mu(M) = 1$ and $\nu \in \mathrm{Haar}(V^*)$ be the Haar measure satisfying $\nu(P) = 1$. Define an isomorphism $\mathrm{Haar}(V^*) \to \mathrm{Haar}(V)^*$ by

$$\langle \nu, \mu \rangle = 1$$

Since both spaces are one-dimensional it defines an isomorphism between $\mathrm{Haar}(V^*)$ and $\mathrm{Haar}(V)^*$.
Let $B' = \{e'_1, \ldots, e'_n\}$ be another basis of V and $C' = \{f'_1, \ldots, f'_n\}$ it's dual basis, M', P' the parallelepipeds spanned by these bases $\mu' \in \mathrm{Haar}(V)$, $\nu' \in \mathrm{Haar}(V^*)$ the Haar measures achieving 1 on M',P' respectively. We let

$$\langle \mu', \nu' \rangle' = 1$$

Note that

$$\begin{aligned}&\mu(M') = |\det(M_{B'}^B)|\mu(M) = |\det(M_{B'}^B)| \\ &\nu(P') = |\det(M_{C'}^C)|\nu(P) = |\det((M_B^{B'})^t)| = |\det(M_{B'}^B)|^{-1}\end{aligned}$$

So

$$\mu = |\det(M_{B'}^{B})|\mu'$$
$$\nu = |\det(M_{B'}^{B})|^{-1}\nu'$$

Therefore

$$\langle \mu, \nu\rangle' = \langle |\det(M_{B'}^{B})|\mu', |\det(M_{B'}^{B})|^{-1}\nu'\rangle' = \langle \mu', \nu'\rangle' = 1$$

So $\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle'$ define the same isomorphism and therefore the isomorphism does not depend on the choice of basis.

5. Choose a basis $e_1, \ldots, e_k$ of W and complete it to a basis $e_1, \ldots, e_n$ of V. Define the distribution $\xi \in \mathrm{Dist}(V \setminus W)$ by

$$\langle \xi, f\rangle = \int_{\mathbb{R}^n} e^{e^{\frac{1}{x_n}}} f(x_1e_1 + \cdots + x_ne_n)\, dx_1 \cdots dx_{n-1}\, dx_n$$

i.e. ξ is $e^{e^{\frac{1}{x_n}}} dx$. Since f is compactly supported in $V \setminus W$, then it is well defined ($\exists \epsilon > 0$ s.t. $f(x_1e_1 + \cdots + x_ne_n) = 0\ \forall |x_n| \le \epsilon$). Assume that $\exists \eta \in \mathrm{Dist}(V)$ s.t. $\eta\big|_{V\setminus W} = \xi$. Let $f_m \in C_c^\infty(V)$ be functions, compactly supported on $V \setminus W$ s.t. $f_m \to f$ and f is some non-negative, compactly supported function, exponentially decreasing to 0 at W (so all of its derivations of any order at W are 0, and thus it is in the closure of $C_c^\infty(V \setminus W)$ in $C_c^\infty(V)$). Then

$$\begin{aligned}
\langle \eta, f\rangle &= \lim_{m\to\infty}\langle \eta, f_m\rangle && \eta \text{ is continuous}\\
&= \lim_{m\to\infty}\langle \xi, f_m\rangle && f_m \in C_c^\infty(V \setminus W)\\
&= \lim_{m\to\infty} \textstyle\int_{\mathbb{R}^n} e^{e^{\frac{1}{x_n}}} f_m(x_1e_1 + \cdots + x_ne_n)\, dx_1 \cdots dx_{n-1}\, dx_n\\
&= \textstyle\int_{\mathbb{R}^n} e^{e^{\frac{1}{x_n}}} f(x_1e_1 + \cdots + x_ne_n)\, dx_1 \cdots dx_{n-1}\, dx_n = \infty
\end{aligned}$$

Where the last equality is true since f decreases exponentially and $e^{e^{\frac{1}{x_n}}}$ grows super-exponentially and the functions are non-negative. Therefore there is not $\eta \in \mathrm{Dist}(V)$ s.t. $\eta\big|_{V\setminus W} = \xi$.

6.

$$G_i = \{f \in C_c^\infty(V) \mid Df = 0\ \forall |D| \le i \text{ differntial operation}\}$$

$\Phi : G_{i-1}/G_i \to C_c^\infty(W, \mathrm{Sym}^i(W^\perp))$ is given by

$$\Phi(f)(w)(v_1, \ldots, v_i) = \partial_{v_1} \cdots \partial_{v_i} f\,(w)$$

Where we identify $\mathrm{Sym}^i(W^\perp) = \mathrm{Sym}^i((V/W)^*) = \mathrm{Sym}^i(V/W)^*$ with $\{f : V^i \to \mathbb{R} \text{ symmetric and multilinear} \,|\, f\big|_{W\times V\times V\times\cdots\times V} = 0\}$.

Choose a basis $\{e_1, \ldots, e_k\}$ of W and complete it to a basis of V: $\{e_1, \ldots, e_n\}$.

Let $\varphi \in C_c^\infty(W, \mathrm{Sym}^i(W^\perp))$. For a multi-index $\alpha = (\alpha_{k+1}, \ldots, \alpha_n)$ with $|\alpha| = i$ denote by $e^\alpha = (e_{k+1}, \ldots, e_{k+1}, e_{k+2}, \ldots, e_{k+2}, \ldots, e_n, \ldots, e_n)$ where each e_j apears α_j times. Define $f : V \to \mathbb{R}$ by

$$f(x_1e_1 + \cdots + x_ne_n) := \sum_{\alpha=(\alpha_{k+1},\ldots,\alpha_n)\,|\alpha|=i} \frac{x^\alpha}{\alpha!}\varphi(x_1e_1 + \cdots + x_ke_k)(e^\alpha)$$

Where $x^\alpha = x_{k+1}^{\alpha_{k+1}} \cdots x_n^{\alpha_n}$ and $\alpha! = \alpha_{k+1}! \cdots \alpha_n!$.

Let $\partial_j := \partial_{e_j}$, and for some multi-index $\beta = (\beta_1, \ldots, \beta_n)$, let $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$. Notice that $\partial^\beta f(w) = 0$ for every $|\beta| \le i-1$ and $w \in W$ (since all of the expresseions in the sum contain monomials of degree i in the coeffiecients of basis elements which are not in W). Now, for every $v_1, \ldots, v_r$ with $r < i$, $\partial_{v_1} \cdots \partial_{v_r}$ is linearly dependent on $\{\partial^\beta \,|\, |\beta| \le i-1\}$ and thus $\partial_{v_1} \cdots \partial_{v_r} f(w) = 0$. I.e. $f \in G_{i-1}$.

We want to show that $\varphi = \Phi(f)$. It is sufficient to prove that $\varphi(w)(e^\beta) = \Phi(f)(w)(e^\beta)$ for every $|\beta| = i$. Let $|\beta| = i$. For $\alpha = (\alpha_{k+1}, \ldots, \alpha_n)$ with $|\alpha| = i$, denote $f_\alpha(x_1e_1 + \cdots + x_ne_n) := \frac{x^\alpha}{\alpha!}\varphi(x_1e_1 + \cdots + x_ke_k)(e^\alpha)$, so $f = \sum_{|\alpha|=i} f_\alpha$. $\partial^\beta f_\alpha$ is a linear combination of polynomials in $x_{k+1}, \ldots, x_n$ multiplied by the derivativies of $\varphi(x_1e_1 + \cdots + x_ke_k)(e^\alpha)$. Notice that since $|\beta| = i$, each monomial containing some none-trivial derivation of φ will also be multiplied by a non-trivial monomial in $x_{k+1}, \ldots, x_n$ and thus will be 0 on W. Therefore

$$\partial^\beta f_\alpha(w) = \frac{\partial^\beta(x^\alpha)}{\alpha!}\varphi(w)(e^\alpha)$$

If $\beta \ne \alpha$ then it is 0, otherwise it is exactly $\varphi(w)(e^\beta)$. Thus

$$\Phi(f)(w)(e^\beta) = \partial^\beta f(w) = \sum_{|\alpha|=i} \partial^\beta f_\alpha(w) = \varphi(w)(e^\beta)$$

So $\Phi(f) = \varphi$ and Φ is surjective.