Generalized Functions Exercise 6

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1. Let ω_1 be the set of countable ordinals, endowed with the discrete topology, and let $L \subseteq \omega_1$ be the set of limit countable ordinals. Define $X := [0,1] \times \omega_1 \setminus \{0\} \times L$. We define the requested topological space M by gluing the end-points of successor ordinals. i.e.

$$M = X \big/ (0, \alpha + 1) \sim (1, \alpha)$$

So M is a "really long line".
Let $[x, \alpha] \in M$. If $0 < x < 1$ then $(0,1) \times \{\alpha\}$ is an open neighborhood of $[x, \alpha]$ homeomorphic to $\mathbb{R}$. Otherwise, if $x = 0$ then α is a successor, and whence $[1, \alpha - 1] = [0, \alpha]$ so we may assume $x = 1$, and now $(\frac{1}{2}, 1] \times \{\alpha\} \cup [0, \frac{1}{2}) \times \{\alpha + 1\}$ is an open neighborhood of $[x, \alpha]$ homeomorphic to $\mathbb{R}$. So M is locally homeomorphic to $\mathbb{R}$, and it is obviously Hausdorff.
Define $U_\alpha := \cap_{\beta \leq \alpha} (0, 1] \times \{\beta\} \setminus 1 \times \{\alpha\}$ for $\alpha < \omega_1$. It is an open cover of M which does not admit a locally finite refinement. So M is not paracompact.

2. Let **Vect** denote the category of finite-dimensional real vector spaces endowed with the Euclidean topology. For a smooth manifold or a p-adic analytic manifold M let $\mathbf{VB}(M)$ denote the category of real vector bundles over M.

Definition. *Let $\lambda : \mathbf{Vect}^k \times (\mathbf{Vect}^{op})^l \to \mathbf{Vect}$ be a functor, where $\mathbf{Vect}^{op}$ is the opposite (dual) category of* **Vect**. *We say that λ is topologically-continuous if for any $V_1, \ldots, V_{k+l}, V'_1, \ldots, V'_{k+l} \in \mathbf{Vect}$*

$$\lambda : \prod_{i=1}^{k} Hom(V_i, V'_i) \times \prod_{i=k+1}^{l} Hom(V'_i, V_i) \to Hom(\lambda(V_1, \ldots, V_{k+l}), \lambda(V'_1, \ldots, V'_{k+l}))$$

is continuous as a map of real spaces.

Claim. *Let $\lambda : \mathbf{Vect}^k \times (\mathbf{Vect}^{op})^l \to \mathbf{Vect}$ be a topologically-continuous functor, then for any smooth manifold or a p-adic analytic manifold M,* *there exists a unique functor* $\lambda_M : \mathbf{VB}(M)^k \times (\mathbf{VB}(M)^{op})^l \to \mathbf{VB}(M)$ *s.t.*

$$\big(\lambda_M(E_1, \dots, E_{k+l})\big)_x = \lambda((E_1)_x, \dots, (E_{k+l})_x)$$

for any $E_1, \dots, E_{k+l} \in \mathbf{VB}(M)$*,* $x \in M$*.*

Proof. Let $E_1, \dots, E_{k+l}$ be real vector bundles over M. Define $\Lambda = \lambda_M(E_1, \dots, E_{k+l})$ as the set

$$\{(x, \lambda((E_1)_x, \dots, (E_{k+l})_x) \,|\, x \in M\}$$

with the obvious projection $\pi : \Lambda \twoheadrightarrow M$. For $x \in M$ let U_x be an open neighborhood of x s.t. there exists trivializations $\varphi_{x,i} : E_i\big|_{U_x} \xrightarrow{\sim} U_x \times (E_i)_x$.
For $U \subseteq M$ open, define $\Lambda\big|_U := \pi^{-1}(U)$. Define the set-functions $\xi_x : \Lambda\big|_{U_x} \to U_x \times \lambda((E_1)_x, \dots, (E_{k+l})_x)$ in the following way:
For $y \in U_x$ we have linear isomorhpisms $\varphi_{i,x,y} : (E_i)_y \to (E_i)_x$ given by $v \mapsto \varphi_{x,i}(y, v)$. λ is a functor of linear spaces, so we have a morphism

$$\lambda(\varphi_{1,x,y}, \dots, \varphi_{k+l,x,y}) : \lambda((E_1)_y, \dots, (E_{k+l})_y) \to \lambda((E_1)_x, \dots, (E_{k+l})_x)$$

(which is an isomorphism of linear spaces since each of the above morphisms is an isomorphism). It induces the function $\xi_x : \Lambda\big|_{U_x} \to U_x \times \lambda((E_1)_x, \dots, (E_{k+l})_x)$ given by

$$\xi_x(y, v) = (y, \lambda(\varphi_{1,x,y}, \dots, \varphi_{k+l,x,y})(v))$$

Topologise Λ with the weakest topology s.t. each ξ_x is continuous. i.e. a basis for the topology is

$$\{\xi_x^{-1}(U \times V) \,|\, x \in M,\ U \subseteq M \text{ open},\ V \subseteq \lambda((E_1)_x, \dots, (E_{k+l})_x) \cong \mathbb{R}^d \text{ open}\}$$

Since λ is topologically-continuous, each ξ_x is also a homeomorphism. Now Λ is a topological space with a continuous surjection $\pi : \Lambda \twoheadrightarrow M$ satisfying

(a) $\Lambda_x = \pi^{-1}(x) = \lambda((E_1)_x, \dots, (E_{k+l})_x)$, which is a finite-dimensional real vector space.

(b) For every $x \in M$ there exists an open neighborhood $x \in U_x$ and a trivialization $\xi_x : \Lambda\big|_{U_x} \xrightarrow{\sim} U_x \times \Lambda_x$ which is a homeomorphism satisfying $\pi \circ \xi_x = \pi$.

(c) For every $y \in U_x$ $v \mapsto \xi_x(y, v)$ is a linear isomorphism $\Lambda_y \to \Lambda_x$.

Now, notice that $(-)_M$ is indeed a functor, for if $\eta : \lambda_1 \to \lambda_2$ is a natural transformation, then we can define $\eta_M : (\lambda_1)_M \to (\lambda_2)_M$ locally as the product with the identity. And it is obvious that $(\mathbb{1})_M = \mathbb{1}$ and that $(\eta \circ \eta')_M = \eta_M \circ \eta'_M$. $\square$

(a) Consider the functor $(-)^* : \mathbf{Vect}^{\mathrm{op}} \to \mathbf{Vect}$ and apply the lemma.

(b) Look at the functor $\oplus : \mathbf{Vect}^2 \to \mathbf{Vect}$ and apply the lemma. It is easy that it satisfies the universal property: Let E_1 and E_2 be two real vector bundles. Near every point $x \in M$ there is a neighborhood U_x and trivializations $\varphi_x : E_1\big|_{U_x} \xrightarrow{\sim} U_x \times (E_1)_x$, $\psi_x : E_2\big|_{U_x} \xrightarrow{\sim} U_x \times (E_2)_x$, $\xi_x : E_1 \oplus E_2\big|_{U_x} \xrightarrow{\sim} U_x \times (E_1)_x \oplus (E_2)_x$. Thus locally it obviously satisfies the universal property.
Build morphisms $\iota_j : E_j \to E_1 \oplus E_2$ locally: Near x it will be given by the natural morphism $\iota_j : E_j\big|_{U_x} \to E_1\big|_{U_x} \oplus E_2\big|_{U_x}$. It agrees on the intersection because $(E_1\big|_{U_x} \oplus E_2\big|_{U_x})\big|_{U_y}$ and $(E_1\big|_{U_y} \oplus E_2\big|_{U_y})\big|_{U_x}$ both satisfy the universal property of direct sum of $E_1\big|_{U_x \cap U_y}$ and $E_2\big|_{U_x \cap U_y}$.
In the same way, if there are morphisms $f_1 : E_1 \to G$, $f_2 : E_2 \to G$ for some vector bundle G, there exists a unique morphism $f : E_1 \oplus E_2 \to G$ satisfying $f \circ \iota_j = f_j$.
It is also easy to see that it satisfies the universal propery of a product in this category.

(c) Consider the functor $\otimes : \mathbf{Vect}^2 \to \mathbf{Vect}$ and apply the lemma.

(d) Let $E_1 \subseteq E_2$. Define E_1/E_2 as the set

$$\{(x, (E_1)_x/(E_2)_x) \mid x \in M\}$$

with the obvious projection $\pi : E_1/E_2 \twoheadrightarrow M$. For $x \in M$ let U_x be an open neighborhood of x s.t. there exists trivializations $\varphi_x : E_1\big|_{U_x} \xrightarrow{\sim} U_x \times (E_1)_x$, and $\psi_x := \varphi_x\big|_{E_2} : E_2\big|_{U_x} \xrightarrow{\sim} U_x \times (E_2)_x$, and $(E_2)_x \subseteq (E_1)_x$.
For $U \subseteq M$ open, define $(E_1/E_2)\big|_U := \pi^{-1}(U)$. Define the set-functions $\xi_x : (E_1/E_2)\big|_{U_x} \to U_x \times (E_1)_x/(E_2)_x$ in the following way: For $y \in U_x$ we have linear isomorhpisms $\varphi_x : (E_j)_y \to (E_j)_x$ given by $v \mapsto \varphi_x(y, v)$. We have a morphism

$$\overline{\varphi_x} : (E_1)_x/(E_2)_x(E_1)_y/(E_2)_y$$

which is an isomorphism of linear spaces. It induces the function $\xi_x : (E_1/E_2)\big|_{U_x} \to U_x \times (E_1)_x/(E_2)_x$ given by

$$\xi_x(y, v) = (y, \overline{\varphi_x}(v))$$

Topologise Λ with the weakest topology s.t. each ξ_x is continuous. i Topologise E_1/E_2 with the weakest topology s.t. each ξ_x is continuous. i.e. a basis for the topology is

$$\{\xi_x^{-1}(U \times V) \,|\, x \in M,\, U \subseteq M \text{ open},\, V \subseteq (E_1)_x/(E_2)_x \cong \mathbb{R}^d \text{ open}\}$$

Now E_1/E_2 is a topological space with a continuous surjection $\pi : E_1/E_2 \twoheadrightarrow M$ satisfying

i. $(E_1/E_2)_x = \pi^{-1}(x) = (E_1)_x/(E_2)_x$ which is a finite-dimensional real vector space.

ii. For every $x \in M$ there exists an open neighborhood $x \in U_x$ and a trivialization $\xi_x : (E_1/E_2)\big|_{U_x} \xrightarrow{\sim} U_x \times (E_1/E_2)_x$ which is a homeomorphism satisfying $\pi \circ \xi_x = \pi$.

iii. For every $y \in U_x$ $v \mapsto \xi_x(y, x)$ is a linear isomorphism $(E_1/E_2)_y \to (E_1/E_2)_x$.

We can show locally that it satisfies the universal condition of cokernel, the same as in the direct sum part.

(e) Look at the functors Λ^k : **Vect** $\to$ **Vect** and Sym^k : **Vect** $\to$ **Vect**, and apply the lemma.

(f) Look at the functor Dens : **Vect** $\to$ **Vect** and apply the lemma.

3. (a) We gave 3 constructions to the tangent space at a point x of a smooth manifold M:

i. $T_xM =$ The set of smooth curves $\alpha : (-\epsilon, \epsilon) \to M$ s.t. $\alpha(0) = x$ under the identification of 2 curves α, β if for some chart (and thus for all charts) $\varphi : x \in U \subseteq M \hookrightarrow \mathbb{R}^m$ near x, $(\varphi\alpha)'(0) = (\varphi\beta)'(0)$.

We give it a vector space structure by

$$a[\alpha] + b[\beta] := [\varphi^{-1}(a\varphi\alpha + b\varphi\beta)]$$

This structure does not depend on the chart: Assume $\psi : x \in V \subseteq M \hookrightarrow \mathbb{R}^m$ is another chart. Wlog $U = V$, so $\varphi\psi^{-1}$ is a smooth map of open subsets in $\mathbb{R}^n$. (U, φ) is a chart so to show that the structure is unique it is enough to show that

$$(\varphi(\varphi^{-1}(a\varphi\alpha + b\varphi\beta)))'(0) = (\varphi(\psi^{-1}(a\psi\alpha + b\psi\beta)))'(0)$$

$$\begin{aligned}
\varphi(\psi^{-1}(a\psi\alpha + b\psi\beta))'(0) &= D_x(\varphi\psi^{-1})(a(\psi\alpha)'(0) + (b\psi\beta)'(0)) \\
&= aD_x(\varphi\psi^{-1})(\psi\alpha)'(0) + bD_x(\varphi\psi^{-1})(\psi\beta)'(0) \\
&= a(\varphi\alpha)'(0) + b(\varphi\beta)'(0) \\
&= (\varphi(\varphi^{-1}(a\varphi\alpha + b\varphi\beta)))'(0)
\end{aligned}$$

ii. The set of derivations at x on the global sections of M:

$$\mathrm{Der}_x(M) = \{\delta : C^\infty(M) \to \mathbb{R} \mid \text{ s.t. } \delta(fg) = f(x)\delta g + g(x)\delta f\}$$

iii. $\left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$ where $\mathfrak{m}_x = \{f \in C^\infty(M) \mid f(x) = 0\}$.

Choose local coordinates $\varphi : U \to U'$ where $x \in U \subseteq M$ open and $U' \subseteq \mathbb{R}^m$ open for M around x. We define a morphism $D : T_xM \to \mathrm{Der}_x(M)$: For $[\alpha] \in T_xM$ and $f \in C^\infty(M)$

$$D([\alpha])(f) := \frac{d}{dt}\Big|_{t=0}(f\alpha) = d(f\varphi^{-1})((\varphi\alpha)'(0))$$

D obviuosly does not depend on the choice of local coordinates.
D is well defined: Assume $\alpha \sim \beta$ then $(\varphi\alpha)'(0) = (\varphi\beta)'(0)$ so $D([\alpha])(f) = D([\beta])(f)$.
D is linear: Let $[\alpha], [\beta] \in T_xM$ and $a, b \in \mathbb{R}$, then

$$\begin{aligned}
D(a[\alpha] + b[\beta]) &= D([\varphi^{-1}(a\varphi\alpha + b\varphi\beta)]) \\
&= d(f\varphi^{-1})((\varphi\varphi^{-1}(a\varphi\alpha + b\varphi\beta))'(0)) \\
&= d(f\varphi^{-1})((a\varphi\alpha + b\varphi\beta)'(0)) \\
&= d(f\varphi^{-1})(a(\varphi\alpha)'(0) + b(\varphi\beta)'(0)) \\
&= ad(f\varphi^{-1})((\varphi\alpha)'(0)) + bd(f\varphi^{-1})((\varphi\beta)'(0)) \\
&= aD([\alpha]) + bD([\beta])
\end{aligned}$$

D is injective: If $D([\alpha]) = 0$ then $d(f \circ \varphi^{-1})((\varphi \circ \alpha)'(0)) = 0$ for any f, so $(\varphi \circ \alpha)'(0) = 0$ and whence $\alpha \sim 0$ so D is injective.
Let $\delta : C^\infty(M) \to \mathbb{R}$ be a derivation at x. Using the local coordinates U at x and multiplying by a bump function we get a derivation $\delta : C^\infty(U \subseteq \mathbb{R}^m) \to \mathbb{R}$. First we may assume wlog that x is 0 in those local coordinates and that U is the open ball around the origin $B = B(0, 1)$.
Notice that $\delta(1) = \delta(1 \cdot 1) = 1 \cdot \delta(1) + 1 \cdot \delta(1) = 2\delta(1)$ so $\delta(1) = 0$. Since δ is linear, $\delta(c) = 0$ for every constant function c. Let $x_i : B \to \mathbb{R}$ be the i-th coordinate function, and write $c_i := \delta x_i \in \mathbb{R}$.
For a smooth function $f : B \to \mathbb{R}$ and a point $y \in B$,

$$f(y) = f(0) + \int_0^1 \frac{df}{dt}(ty)\, dt = f(0) + \sum_{i=1}^m \int_0^1 \frac{\partial f}{\partial x_i}(ty) \cdot y_i \, dt =$$

$$f(0) + \sum_{i=1}^m y_i \int_0^1 \frac{\partial f}{\partial x_i}(ty)\, dt$$

Denote by $g_i(y) := \int_0^1 \frac{\partial f}{\partial x_i}(ty)\,dt$, then $f = f(0) + \sum_{i=1}^m x_i g_i$. Now

$$\delta f = \delta(f(0)) + \sum_{i=1}^m \delta(x_i g_i) = \sum_{i=1}^m (x_i\big|_0 \cdot \delta g_i + c_i g_i(0)) = \sum_{i=1}^m c_i \frac{\partial f}{\partial x_i}(0)$$

Let $\alpha : (-\epsilon, \epsilon) \to M$ be written in local coordinates by $\alpha(t) := \sum_{i=1}^m c_i t$, then in local coordinates

$$D([\alpha])(f) = df(\alpha'(0)) = \langle \nabla f, (c_i)_{i=1}^m \rangle = \sum_{i=1}^m c_i \frac{\partial f}{\partial x_i} = \delta f$$

So $\delta = D([\alpha])$.

Now let $\mathrm{res} : \mathrm{Der}_x(M) \to \mathrm{Hom}(\mathfrak{m}_x, \mathbb{R})$ be given by

$$(\mathrm{res}\delta)(f) := \delta f$$

Notice the if $f \in \mathfrak{m}_x^2$, then f has the form $f = \sum_i g_i h_i$ where $g_i, h_i \in \mathfrak{m}_x$, and whence

$$(\mathrm{res}\delta)(f) = \delta f = \sum \delta(g_i h_i) = \sum (g_i(x)\delta h_i + h_i(x)\delta g_i) = 0$$

Thus we may define $\pi : \mathrm{Der}_x(M) \to \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$ by

$$\pi(\delta)(f + \mathfrak{m}_x^2) := \delta f$$

It is one-to-one, because if $\pi(\delta) = 0$, then for any $f \in C^\infty(M)$, $f - f(x) \in \mathfrak{m}_x$, so

$$\delta f = \delta(f - f(x) + f(x)) = \delta(f - f(x)) + \delta(f(x)) = \pi(\delta)(f - f(x) + \mathfrak{m}_x^2) = 0$$

and whence $\delta = 0$.
Let $\varphi \in \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$. We define $\delta : C^\infty(M) \to \mathbb{R}$ by

$$\delta f := \varphi(f - f(x) + \mathfrak{m}_x^2)$$

It is obviously linear. Let $f, g \in C^\infty(M)$, then

$$\begin{aligned}
\delta(fg) &= \varphi(fg - f(x)g(x) + \mathfrak{m}_x^2) \\
&= \varphi(fg - f(x)g(x) - (f - f(x))(g - g(x)) + \mathfrak{m}_x^2) \\
&= \varphi(f(x)(g - g(x)) + \mathfrak{m}_x^2) + \varphi(g(x)(f - f(x)) + \mathfrak{m}_x^2) \\
&= f(x)\delta g + g(x)\delta f
\end{aligned}$$

so δ is a derivation, and it is obvious that $\pi(\delta) = \varphi$. So π is surjective.

$$T_x M \xrightarrow{D} \mathrm{Der}_x(M) \xrightarrow{\pi} \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$$

are canonical isomorphisms and thus all the definitions are equivalent.

(b) Let $T : (C^\infty)_* \to \mathbf{Vect}$, where $(C^\infty)_*$ is the category of based smooth real manifolds, be the functor given by

$$T(M, x) := T_x M$$

For $f : (M, x) \to (N, y)$, we define

$$T_x f = f_* : T_x M \to T_y N$$

by $f_*([\alpha]) = [f\alpha]$. It is well defined: Assume that $\alpha \sim \beta$, then for some chart $\varphi : x \in U \hookrightarrow \mathbb{R}^m$, $(\varphi\alpha)'(0) = (\varphi\beta)'(0)$. Let $\psi : y \in V \subseteq N \hookrightarrow \mathbb{R}^n$ be a chart of N. Denote by $F := \psi f \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$. Then

$$(\psi f \alpha)'(0) = \big(F \circ (\varphi\alpha)\big)'(0) = D_{\varphi(x)} F \cdot (\varphi\alpha)'(0) =$$

$$D_{\varphi(x)} F \cdot (\varphi\beta)'(0) = \big(F \circ (\varphi\beta)\big)'(0) = (\psi f \beta)'(0)$$

So $f\alpha \sim f\beta$.
Now it obvious that $\mathbb{1}_* = \mathbb{1}$ and that $(g \circ f)_* = g_* \circ f_*$ so T is indeed a functor.

i. Let V be a real vector space, and look at $(V, 0) \in (C^\infty)_*$. Choose a basis $v_1, \dots, v_n$ for V. Let $\varphi : V \to \mathbb{R}^n$ be the morphism given by $v_i \mapsto e_i$. φ is a diffeomorphism (and in particular a chart). We define a function $L : T_0 V \to V$ by

$$L([\alpha]) = \varphi^{-1}((\varphi\alpha)'(0))$$

And it is exactly the definition we gave for the vector space structure on $T_0 V$ so it is well defined isomorphism which does not depend on the chart (and in particular on the basis of V). So $T_0 V = V$.

ii. Assume $f, g : (M, x) \to (\mathbb{R}, 0)$ s.t. in local coordinates around x, $(f_1 - f_2)(y) = o(|y - x|)$, i.e. in local coordinates $\nabla_0 f_1 = \nabla_0 f_2$. Let $[\alpha] \in T_x M$. Then in local coordinates

$$T_x f_1([\alpha]) = \nabla_0 f_1 \cdot \alpha'(0) = \nabla_0 f_2 \cdot \alpha'(0) = T_x f_2([\alpha])$$

So $T_x f_1 = T_x f_2$.

iii. Let $U \subseteq M$ be an open submanifold. Since our definition of tangent space is local it is obvious that $T_xU = T_xM$ for any $x \in U$.

Thus T abide the axioms of a tangent space.

4. Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^k)$, then for any $\mu \in C^\infty(\mathbb{R}^k)$, $f^*\mu = \mu \circ f$ is smooth. So we have an inclusion

$$C^\infty(\mathbb{R}^n, \mathbb{R}^k) \subseteq \{f : \mathbb{R}^n \to \mathbb{R}^k \,|\, f^*\mu \in C^\infty(\mathbb{R}^n)\, \forall \mu \in C^\infty(\mathbb{R}^k)\}$$

On the other hand, let $f : \mathbb{R}^n \to \mathbb{R}^k$ s.t. $f^*\mu \in C^\infty(\mathbb{R}^n)\, \forall \mu \in C^\infty(\mathbb{R}^k)$. i.e. for any smooth $\mu : \mathbb{R}^k \to \mathbb{R}$, $\mu \circ f$ is also smooth. In particular $\pi_j \circ f$ is smooth, where $\pi_j : \mathbb{R}^k \to \mathbb{R}$ is the projection on the j-th coordinate. Therefore f is smooth. Whence we have an equality

$$C^\infty(\mathbb{R}^n, \mathbb{R}^k) = \{f : \mathbb{R}^n \to \mathbb{R}^k \,|\, f^*\mu \in C^\infty(\mathbb{R}^n)\, \forall \mu \in C^\infty(\mathbb{R}^k)\}$$

5. Let $\phi : M \to N$ be a smooth map between manifolds.

   (a) Assume that ϕ is a closed embedding, then it is an injective immersion by definition. Let $K \subseteq N$ be a compact subset, then $\phi^{-1}(K) = K \cap M$ is a closed subset of K and hence compact. So ϕ is proper.
   Assume now that ϕ is a proper injective immersion. Since ϕ is an injective immersion, $\phi(M)$ is a submanifold of N, diffeomorphic to M. Let $y_n \in \phi(M)$ s.t. $y_n \to y$ for some $y \in N$. Let U be a compact neighborhood of x, then $\phi^{-1}(U)$ is open and, since ϕ is proper, its closure in M is compact. U contains almost all y_n, so for almost all n, $\exists x_n \in \phi^{-1}(U)$ s.t. $\phi(x_n) = y_n$. Since $\overline{\phi^{-1}(U)}$ is compact, there exists a subseries x_{n_k} s.t. $x_{n_k} \to x \in M$. Since ϕ is continuous, $y_{n_k} = \phi(x_{n_k}) \to \phi(x)$ so $\phi(x) = y$ and in particular $y \in \phi(M)$. Therefore $\phi(M)$ is closed.

6. Look at $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(t) = (\cos t, \sin t)$$

Then at a point t, $f'(t) = (-\sin t, \cos t) \neq 0$ so it is an immersion. Yet f in not injective because $f(t + 2\pi) = f(t)$ for any t.