Generalized Functions Exercise 5

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1. G is a locally compact abelian group. For an abelian group H and a function $f : H \to \mathbb{C}$ we define

$$sh_h(f)(x) := f(x - h)$$

For a measure η on G and $g \in G$ we define

$$(sh_g\eta)(A) := \eta(g + A)$$

(a)

Lemma. *$\int_G f\, d(sh_g\eta) = \int_G sh_g f\, d\eta$ for any measurable f and $\eta \in \mu_c(G)$.*

Proof. Let $A \subseteq G$ be a Borel set, and look at the indicator function 1_A.

$$\int_G 1_A\, d(sh_g\eta) = sh_g\eta(A) = \eta(g + A) = \int_G 1_{g+A}\, d\eta = \int_G sh_g(1_A)\, d\eta$$

So the claim is true for indicator functions. By linearity it is true for simple functions. Since simple functions are dense in the measurable functions, and $\int_G d(sh_g\eta)$ and $\int_G d\eta$ are continuous, the claim follows. □

Let $\eta \in \mu_c(G)$, $g \in G$ and $\chi \in G^\vee$.

$$\begin{aligned}\mathcal{F}(sh_g\eta)(\chi) &= \textstyle\int_G \overline{\chi}\, d(sh_g\eta) = \int_G sh_g\overline{\chi}\, d\eta \\ &= \textstyle\int_G \overline{\chi}(-g)\overline{\chi}\, d\eta = \chi(g)\mathcal{F}(\eta)(\chi)\end{aligned}$$

(b) Let $\eta \in \mu_c(G)$ and $\chi, \tau \in G^\vee$.

$$\begin{aligned}\mathcal{F}(\chi\eta)(\tau) &= \textstyle\int_G \overline{\tau}\, d(\chi\eta) = \int_G \overline{\tau}\chi\, d\eta \\ &= \mathcal{F}(\eta)(\tau\overline{\chi}) = sh_\chi(\mathcal{F}(\eta))\,(\tau)\end{aligned}$$

2. Let F be a non-archemedian field. Let $f \otimes h \in S(V) \otimes \mathrm{Haar}(V) = S(V, \mathrm{Haar}(V))$. We wish to show that

$$\begin{array}{rcl} \mathcal{F}(f \otimes h) & : & V^\vee \to \mathbb{C} \\ & & \chi \mapsto \int_V f\overline{\chi}\, dh \end{array}$$

is smooth and compactly supported.

$f \in S(V)$, so it is smooth, i.e. locally constant. For each $x \in \mathrm{supp}(f)$ let H_y be a compact open subgroup of V s.t. $f\big|_{y+H_y}$ is constant (it exists because compact-open subgroups are a basis for the topology at the origin). Since $\mathrm{supp}(f)$ is compact and $\mathrm{supp}(f) \subseteq \bigcup(y+H_y)$, $f = \sum \alpha_i 1_{y_i+B_{y_i}}$ for some finitely many y_i. Therefore we may assume that $f = 1_{y+H}$ for some compact open subgroup of V and some $y \in V$. Now the Fourier transform is of the form

$$\begin{array}{rcl} \mathcal{F}(f \otimes h) & : & V^\vee \to \mathbb{C} \\ & & \chi \mapsto \int_{y+H} \overline{\chi}\, dh(x) \end{array}$$

Let $\chi \in V^\vee$.

$$\int_{y+H} \overline{\chi}(x)\, dh(x) = \int_H \overline{\chi}(x+y)\, dh(x+y) = \chi(-y)\int_H \overline{\chi}\, dh(x)$$

So we may assume $y = 0$.

Let $J \subseteq S^1$ be an open interval containing 1 and not containing any half circle. Look at the open subset $U = U(H,J) = \{\chi \in V^\vee \mid \chi(H) \subseteq J\}$. Let $\chi \in U$. Then $\chi\big|_H : H \to S^1$ is a homomorphism, and thus its image is a subgroup of S^1. Since its image lies in a small interval it must be the trivial subgroup and hence $U = \{\chi \in V^\vee \mid \chi(H) = 1\}$ and thus it is also compact.

Let $\eta \in V^\vee$, then ηU is an open neighborhood of η. Let $\chi \in U$, then

$$\mathcal{F}(f \otimes h)(\eta\chi) = \int_H \overline{\eta}\,\overline{\chi}\, dh = \int_H \overline{\eta}\, dh = \mathcal{F}(f \otimes h)(\eta)$$

i.e. $\mathcal{F}(f\otimes h)\big|_{\eta U} = \mathcal{F}(f\otimes h)(\eta)$ is constant, so $\mathcal{F}(f\otimes h)$ is locally constant, i.e. smooth.

In order to prove that it is compactly supported we will use the following lemma:

Lemma. *Let G be a comapct group, μ a non-trivial Haar measure on G and χ a non-trivial character of G. Then*

$$\int_G \chi\, d\mu = 0$$

Proof. Since χ is non-trivial, there exists a $g \in G$ s.t. $\chi(g) \neq 1$.

$$\int_G \chi \, d\mu = \int_G \chi \, d(sh_g \mu) = \int_G sh_g \chi \, d\mu = \int_G \chi(-g)\chi \, d\mu = \chi(-g) \int_G \chi \, d\mu$$

and thus $\int_G \chi \, d\mu = 0$ as required. $\square$

Now, let $\chi \in V^\vee$ s.t. $\mathcal{F}(f \otimes h)(\chi) \neq 0$. H is a compact group, $h\big|_H$ is a Haar measure on H and $\chi\big|_H$ is a character of H, and by the lemma, since $\mathcal{F}(f \otimes h)\chi = \int_H \overline{\chi} \, dh \neq 0$, $\overline{\chi}\big|_H$ is the trivial character (and hence $\chi\big|_H = 1$), i.e. $\chi \in U$.

Therefore $\{\chi \in V^\vee \,|\, \mathcal{F}(f \otimes h)(\chi) \neq 0\} \subseteq U$. Since U is closed $\mathrm{supp}(f) = \overline{\{\chi \in V^\vee \,|\, \mathcal{F}(f \otimes h)(\chi) \neq 0\}} \subseteq U$. Since U is compact and $\mathrm{supp}(f)$ is closed, $\mathrm{supp}(f)$ is compact.

3. We identify $V^\vee$ with the linear space of distributions, i.e. continuous homomorphisms $\xi : V \to \mathbb{R}/\mathbb{Z}$ by $\xi \mapsto e^{2\pi i \xi}$.

Claim. *Let $\alpha \in Haar(V^\vee)$, $\beta \in Haar(V)$ and $g \in C_c^\infty(V^\vee)$ s.t. $g(0) = 0$, then*

$$\langle \mathcal{F}(g \cdot \alpha), \beta \rangle = 0$$

Proof. Choose a basis $e_1, \ldots, e_n$ for $V^\vee$. Thus we may write for $\xi = \sum_{j=1}^n \xi_j e_j$: $g(\xi) = \sum_{j=1}^n \xi_j g_j(\xi)$ where $g_j(\xi) = \int_0^1 \frac{\partial g}{\partial \xi_j}(t\xi)$. Using this basis $\alpha = c \, d\xi_1 \cdots d\xi_n$.

We notice that for a function $f \in C_c^\infty(V^\vee)$,

$$\begin{aligned}
\mathcal{F}(\xi_j f \alpha)(x) &= \textstyle\int_{V^\vee} e^{2\pi i \xi(x)} \, d(\xi_j f \alpha)(\xi) \\
&= c \textstyle\int_{V^\vee} e^{2\pi i \xi(x)} \xi_j f(\xi) \, d\xi_1 \cdots d\xi_n \\
&= c \textstyle\int_{V^\vee} f(\xi) \frac{1}{2\pi i} \frac{\partial}{\partial \xi_j} e^{2\pi i \xi}(x) \, dx_1 \cdots dx_n \\
&= -\tfrac{c}{2\pi i} \textstyle\int_{V^\vee} \frac{\partial f}{\partial \xi_j}(\xi) e^{2\pi i \xi(x)} \, dx_1 \cdots dx_n \\
&= \tfrac{i}{2\pi} \textstyle\int_{V^\vee} e^{2\pi i \xi(x)} \, d(\frac{\partial f}{\partial \xi_j} \alpha)(\xi) \\
&= \tfrac{i}{2\pi} \mathcal{F}(\tfrac{\partial f}{\partial \xi_j} \alpha)(\xi)
\end{aligned}$$

Thus

$$\begin{aligned}
\langle \mathcal{F}(g \cdot \alpha), \beta \rangle &= \langle \mathcal{F}(\textstyle\sum_{j=1}^n \xi_j g_j \cdot \alpha), \beta \rangle \\
&= \textstyle\sum_{j=1}^n \langle \mathcal{F}(\xi_j g_j \cdot \alpha), \beta \rangle \\
&= \tfrac{i}{2\pi} \textstyle\sum_{j=1}^n \langle \frac{\partial}{\partial \xi_j} \mathcal{F}(g_j \cdot \alpha), \beta \rangle = 0
\end{aligned}$$

$\square$

Now we define a morphism $\mathrm{Haar}(V^\vee) \to \mathrm{Haar}(V)^*$ be

$$\langle \alpha, \beta \rangle := \langle \mathcal{F}(f \cdot \alpha), \beta \rangle$$

for all $\alpha \in \mathrm{Haar}(V^\vee)$, $\beta \in \mathrm{Haar}(V)$ and some $f \in C_c^\infty(V^\vee)$ s.t. $f(0) = 1$. By the lemma this action does not depend on the choice of f so it is well defined. Since both spaces are one-dimensional, the associated morphism is an isomorphism.

4. Let $\xi \in S^*(V)$. $\mathcal{F}_0 : S^*(V) \to S^*(V^\vee, \mathrm{Haar}(V^\vee))$ is the dual of the fourier transform of Schwartz functions $\mathcal{F} : S(V^\vee, \mathrm{Haar}(V^\vee)) \to S(V)$ given by

$$\mathcal{F}(\alpha)\,(x) = \int_{V^\vee} \chi(x)\, d\alpha(\chi)$$

for $\alpha \in S(V^\vee, Haar(V^\vee))$. Thus

$$\langle \mathcal{F}_0(\xi), \alpha \rangle = \langle \xi, \mathcal{F}(\alpha) \rangle$$

For any $\alpha \in S^(V^\vee, \mathrm{Haar}(V^\vee))$.

Choose a non-trivial Haar measure $\mu \in \mathrm{Haar}(V)$ and its dual Haar measure $\nu \in \mathrm{Haar}(V^\vee)$, i.e. $\langle \mu, \nu \rangle = 1$. We identify each $\alpha = f \cdot \nu \in S(V^\vee, \mathrm{Haar}(V^\vee))$ with $f \otimes \nu \in S(V^\vee) \otimes \mathrm{Haar}(V^\vee)$ (and by the same way $S(V, \mathrm{Haar}(V)) \cong S(V) \otimes \mathrm{Haar}(V^\vee)$. We also identify

$$S^*(V^\vee, \mathrm{Haar}(V^\vee)) \cong S^*(V^\vee) \otimes \mathrm{Haar}(V^\vee)^* \cong S^*(V^\vee) \otimes \mathrm{Haar}(V)$$
$$S^*(V, \mathrm{Haar}(V)) \cong S^*(V) \otimes \mathrm{Haar}(V)^* \cong S^*(V) \otimes \mathrm{Haar}(V^\vee)$$

Now, look at

$$\mathcal{F}_0 \otimes \mathbb{1} : S^*(V^\vee) \otimes \mathrm{Haar}(V) \to S^*(V, \mathrm{Haar}(V)) \otimes \mathrm{Haar}(V) \cong S^*(V) \otimes \mathrm{Haar}(V^\vee) \otimes \mathrm{Haar}(V)$$

Look at the trace morphism $\mathrm{tr} : S^*(V) \otimes \mathrm{Haar}(V^\vee) \otimes \mathrm{Haar}(V) \to S^*(V)$ given by $\mathrm{tr}(\eta \otimes \alpha \otimes \beta) = \langle \alpha, \beta \rangle \cdot \eta$.

$\mathcal{F}_1 : S^*(V^\vee, \mathrm{Haar}(V^\vee)) \to S^*(V)$ is given by $\mathrm{tr} \circ (\mathcal{F}_0 \otimes \mathbb{1})$. Write $\mathcal{F}_0(\xi) = \mathcal{F}_0'(\xi) \otimes \mu$ for $\xi \in S^*(V)$ and $\mathcal{F}_0(\xi) = \mathcal{F}_0'(\xi) \otimes \nu$ for $\xi \in S^*(V^\vee)$. I.e. $\mathcal{F}_0' : \S^*(V) \to S^*(V^\vee)$ is defined by $\langle \mathcal{F}_0'(\xi), g \rangle = \langle Fcal_0(\xi), g\mu \rangle$. Now, for

$f \in S(V)$:

$$\begin{aligned}
\langle \mathcal{F}_1\mathcal{F}_0(\xi), f\rangle &= \langle \mathcal{F}_1(\mathcal{F}_0'(\xi)\otimes\mu), f\rangle \\
&= \langle \mathrm{tr}(\mathcal{F}_0'\otimes \mathbb{1}(\mathcal{F}_0(\xi)\otimes\mu)), f\rangle \\
&= \langle \mathrm{tr}(\mathcal{F}_0(\mathcal{F}_0'(\xi)\otimes\mu)), f\rangle \\
&= \langle \mathrm{tr}(\mathcal{F}_0'(\mathcal{F}_0'(\xi)\otimes\mu\otimes\nu, f\rangle \\
&= \langle \mathcal{F}_0'(\mathcal{F}_0'(\xi)), f\rangle \\
&= \langle \mathcal{F}_0(\mathcal{F}_0'(\xi)), f\mu\rangle \\
&= \langle \mathcal{F}_0'(\xi), \mathcal{F}(f\mu)\rangle \\
&= \langle \mathcal{F}_0(\xi), \mathcal{F}(f\mu)\nu\rangle \\
&= \langle \xi, \mathcal{F}(\mathcal{F}(f\mu)\nu)\rangle
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(\mathcal{F}(f\mu)\nu)(x) &= \textstyle\int_{V^\vee} \overline{\chi}(x)\mathcal{F}(f\mu)(\chi)\,d\nu(\chi) \\
&= \textstyle\int_{V^\vee} \chi(x)\int_V f(y)\overline{\chi}(y)\,d\mu(y)d\nu(\chi) \\
&= \textstyle\int_{V^\vee}\int_V f(y)\overline{\chi}(x+y)\,d\mu(y)d\nu(\chi) \\
&= \textstyle\int_{V^\vee}\int_V f(y-x)\overline{\chi}(y)\,d\mu(y)d\nu(\chi) \\
&= \textstyle\int_{V^\vee}\int_V sh_x f(y)\overline{\chi}(y)\,d\mu(y)d\nu(\chi) \\
&= \textstyle\int_{V^\vee} \mathcal{F}(sh_x f\mu)(\chi)\,d\nu(\chi) \\
&= \langle \mathcal{F}(sh_x f\mu), \nu\rangle \\
&= sh_x f(0)\langle\mu,\nu\rangle = sh_x f(0) = f(-x) = \left(f\circ ()^{-1}\right)(x)
\end{aligned}$$

Where the equalities in the last row are true by the definition of the isomorphism $\mathrm{Haar}(V^\vee)\cong\mathrm{Haar}(V)^*$ and by the choice of μ and ν. Thus

$$\langle \mathcal{F}_1\mathcal{F}_0(\xi), f\rangle = \langle \xi, f\circ ()^{-1}\rangle = \langle \xi^{fl}, f\rangle$$

5. V is a one-dimensional vector space over $\mathbb{R}$ with a positive structure P.

   (a) Remember that

   $$|V| = \{\xi : V^* \to \mathbb{R} \,|\, \xi(\lambda\varphi) = |\lambda|\xi(\varphi)\ \forall\lambda\in\mathbb{R}, \varphi\in V^*$$

   We define a morphism $\xi : V \to |V|$ by

   $$\xi_v(\varphi) = \begin{cases} \varphi(v), & v\in P \\ -\varphi(v), & v\notin P \end{cases}$$

   which is a canonical isomorphism.

(b) Remember that for $\alpha \in \mathbb{Q}$

$$V^\alpha = \{\xi : V^* \to \ | \, \xi(\lambda\varphi) = |\lambda|^\alpha \xi(\varphi) \ \forall \lambda \in \mathbb{R}, \varphi \in V^*\}$$

We define a morphism $V^\alpha \otimes V^\beta \to V^{\alpha+\beta}$ by $\xi_1 \otimes \xi_2 \mapsto \xi_1 \cdot \xi_2$. It is well defined:

$$(\xi_1\xi_2)(\lambda\varphi) = \xi_1(\lambda\varphi)\xi_2(\lambda\varphi) = |\lambda|^\alpha \xi_1(\varphi) \cdot |\lambda|^\beta \xi_2(\varphi) = |\lambda|^{\alpha+\beta}(\xi_1\xi_2)(\varphi)$$

and thus it is a canonical isomorphism.