Generalized functions - Exercise 1

Solve the following exercises. Questions marked with (*) are optional. For a good resource try to use Terence Tao's notes, available at
https://terrytao.wordpress.com/2009/04/19/245c-notes-3-distributions/.

1. Let $f \in L^1_{\text{loc}}(\mathbb{R})$, show that ξ_f defined by $\langle \xi_f, g\rangle = \int\limits_{-\infty}^{\infty} fg dx$ for $g \in C_c^\infty(\mathbb{R})$ is a distribution.

2. Let $U_1, U_2 \subseteq \mathbb{R}$ be open sets and $g \in C_c^\infty(U_1 \cup U_2)$, find a partition of unity of g, i.e. functions $f_1 \in C_c^\infty(U_1)$ and $f_2 \in C_c^\infty(U_2)$ such that $f_1 + f_2 \equiv 1$ in $(U_1 \cup U_2) \cap \text{supp}(g)$. Try to use Exercise 1 (iii) of Terence Tao's notes.

3. Show that there exists a canonical isomorphism $\overline{C_c^\infty(\mathbb{R})}^w \simeq (C_c^\infty(\mathbb{R}))^*$, where the convergence is w.r.t. the weak convergence defined in class.

4. Let ξ_1 and ξ_2 be distributions. Show that,
   a) $\text{supp}(a\xi_1 + b\xi_2) \subseteq \text{supp}(\xi_1) \cup \text{supp}(\xi_1)$.
   b)* $\text{supp}(\xi_1) - \text{supp}(\xi_1)^\circ \subseteq \text{supp}(\xi_1') \subseteq \text{supp}(\xi_1)$.

5. Let ξ be a compactly supported distribution and $f \in C^\infty(\mathbb{R})$, show that $\xi * f$ is smooth.

6. * Show that convolution of distributions is associative, that is for $\xi_1, \xi_2, \xi_3 \in C_c^\infty(\mathbb{R})^*$ we have that $\xi_1 * (\xi_2 * \xi_3) = (\xi_1 * \xi_2) * \xi_3$.

7. (Exercise 4 (i) of Terence Tao's notes.) Recall that for $f \in C_c^k(\mathbb{R})$, we define the C^k norm by $\|f\|_{C^k} = \sup\limits_{x\in\mathbb{R}} \sum\limits_{j=0}^{k} |f^(i)(x)|$. Show that for a compact $K \subseteq \mathbb{R}$ the functional $\xi : C_c^\infty(K) \to \mathbb{R}$ is continuous if and only if there exists $k \geq 0$ and $C > 0$ such that for all $f \in C_c^\infty(K)$

$$|\langle \xi, f\rangle| \leq C\|f\|_{C^k}.$$

8. a) Let A be a differential operator with constant coefficients. Describe the Green function (G_A such that $A(G_A) = \delta_0$) without using generalized functions.
   b) Set,

$$A_{G_A}(g)(y) = \int\limits_{-\infty}^{\infty} G_A(x, y)g(x)dx.$$

   Show that $A(A_{G_A}(g)) = g$ for every $g \in C_c^\infty(\mathbb{R})$.

Generalized functions - Exercise 2

Solve the following exercises. Questions marked with (*) are optional.

1. Show that for a locally convex, complete topological vector space V the following three conditions are equivalent, thus each implying that V is a Fréchet space.

   (a) V is metrizable.

   (b) V is first countable.

   (c) There is a countable collection of semi-norms $\{n_i\}_{i\in\mathbb{N}}$ that defines the basis for the topology over V.

   (Hint: show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).)

2. Let V be a topological vector space, show that for every neighborhood U of 0 there exists an open balanced set W such that $0 \in W \subseteq U$, thus finishing the exercise from class.

3. Let $0 \in C$ be an open convex set in a topological vector space V. Show that if C is balanced then $N_C(x)$ is a semi-norm (note that in class we said C need only be open convex to be a semi-norm, this is false, since every open neighborhood of 0 contains a balanced set, we are OK).

4. Find a locally convex topological vector space V such that V has no continuous norm on it. (Hint: we know semi-norms correspond to convex balanced sets. When can a semi-norm be injective?)

5. Let $W \subseteq V$ be locally convex topological vector spaces, and set V' and W' to be the continuous duals of V and W respectively, and let $()^*$ denote the usual dual.

   (a) Show that the restriction map $V^* \to W^*$ is onto.

   (b) Show that the restriction map $V' \to W'$ is onto.

6. Show that $C^\infty(\mathbb{R})$ is a complete space. (Hint: show it is locally convex and first countable and use the fact that for these kind of topological vector spaces completeness is equivalent to sequential completeness, thus it is enough to show each Cauchy sequence converges.)

7. Show that $f_n \in C_c^\infty(\mathbb{R})$ converges to f with respect to the topology defined in class if and only if it converges as was defined in the first lecture, i.e,

   (a) There is a compact set $K \subset \mathbb{R}$ s.t. $\mathrm{supp}(f) \bigcup\limits_{n\in\mathbb{N}} \mathrm{supp}(f_n) \subseteq K$.

   (b) For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$.

8. * Let V be a topological vector space, construct its completion. (Hint: you can use the construction in Theorem 5.2 of Treves. You don't have to hand it in but try to get the hang of the proof and the ideas used there, which essentially generalize the construction when one passes from $\mathbb{Q}$ to $\mathbb{R}$ using Cauchy sequences.)

9. * Let V be a locally convex linear topological space. Prove that V is Hausdorff iff $\{0\}$ is a closed set.

Generalized functions - Exercise 3

Solve the following exercises. Questions marked with (*) are optional.

1. Determine if the following statement is true, if so prove it, otherwise find a counterexample. If the functions $\{f_n\}_{n\in\mathbb{N}}$ converge weakly to f, then they converge pointwise to f.

2. Finish the proof of the glueability axiom, thus showing that $C_c^\infty(\mathbb{R})^*$ is a sheaf. Explicitly, show that if $U = \bigcup_{i\in I} U_i$ and $\xi_i \in C_c^\infty(U_i)^*$ such that ξ_i and ξ_j agree on overlaps $U_i \cap U_j$, then there exists $\xi \in C_c^\infty(U)^*$ such that $\xi_{|U_i} = \xi_i$.

3. (a) Show that the space of distributions $(C_c^\infty(\mathbb{R}))^*$ is not complete w.r.t the weak topology.

   (b) *Show that its completion is the space of all linear functionals (not necessarily continuous) equipped with the weak topology.

4. Recall that we defined,

$$V_m(C_c^\infty(\mathbb{R}^n), \mathbb{R}^k) = \{f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^i f}{(\partial x)^i}_{|\mathbb{R}^k} = 0, |i| \leq m\},$$

and

$$F_m((C_c^\infty(\mathbb{R}^n))^*, \mathbb{R}^k) = \{\xi \in (C_c^\infty(\mathbb{R}^n))^* : \xi_{|V_m} = 0\}.$$

   (a) Show that $\bigcap_{m=0}^{\infty} V_m = \overline{C_c^\infty(U)}$.

   (b) Show that $\bigcup_{i=0}^{\infty} F_i \neq C_{\mathbb{R}^k}^\infty(\mathbb{R}^n)^*$.

   (c) Show that F_m is invariant w.r.t. diffeomorphisms of $\mathbb{R}^n$ preserving $\mathbb{R}^k$.

   (d) * Let $U \subseteq \mathbb{R}^n$ be open and $\overline{U}$ compact. Show that for every $\xi \in C_{\mathbb{R}^k}^\infty(\mathbb{R}^n)^*$ there exists $\xi' \in F_m$ such that $\xi_{|U} = \xi'_{|U}$, thus $\bigcup_{m=0}^{\infty} F_m$ covers $C_{\mathbb{R}^k}^\infty(\mathbb{R}^n)^*$ locally.

5. Let V be a topological finite dimensional vector space over a field F. Show that,

$$C_c^\infty(\mathbb{R}^n, V^*)^* \simeq (C_c^\infty(\mathbb{R}^n))^* \otimes_F V.$$

Generalized functions - Exercise 4

Solve the following exercises. Questions marked with (*) are optional.

1. Show that for any $x \in \mathbb{Q}$,

$$|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p = 1.$$

2. Let $B_\epsilon(a) = \{x \in \mathbb{Q}_p : |x-a|_p < \epsilon\}$ be the open p-adic ball of radius epsilon around $a \in \mathbb{Q}_p$.
   (a) $B_\epsilon(a)$ is open by definition, Show that it is also closed.
   (b) Show that every point in $B_\epsilon(a)$ is its center.
   (c) Show that there are countable many open balls which contain 0 in $\mathbb{Q}_p$.

3. Let C be the Cantor set.
   (a) Show $\mathbb{Z}_p \cong C$.
   (b) Show $\mathbb{Q}_p \cong C \backslash \{*\}$.
   (c) What are the cardinalities of $\mathbb{Z}_p$ and $\mathbb{Q}_p$? What are the connected components?

4. Find a compact ℓ-space X and $U \subseteq X$ which is not σ-compact (not the union of countably many compact sets).

5. Let F be a non-archimedean (that is, it has a non-archimedean absolute value) local field (locally compact, non-discrete), show it is an ℓ-space.

6. Let X be an ℓ-space.
   (a) * Show that there exists a basis of open compact sets for X.
   (b) Let $K \subseteq X$ a compact set, and $K \subseteq \bigcup U_\alpha$ an open cover. Show there exists disjoint open compact $\{V_i\}_{i=0}^n$ such that for every i there exists α such that $V_i \subseteq U_\alpha$ and $K \subseteq \bigcup_{i=0}^{n} V_i$.

7. * Every σ-compact, S^1 ℓ-space is homeomorphic to one of the following:
   (a) Countable discrete space.
   (b) Cantor set.
   (c) Cantor set minus a point.

8. * Let X be an ℓ-space, show that $C_c^\infty(X)^*$ is a sheaf.

Generalized functions - Exercise 5

Solve the following exercises. Questions marked with (*) are optional.
Fix $W \subseteq V$ and E to be finite dimensional topological vector spaces.

1. Show that $C^{-\infty}(V) \otimes E \simeq (C_c^{\infty}(V, E^* \otimes \mathrm{Haar}(V)))^*$.
2. Define an embedding $C_c^{\infty}(V, E) \hookrightarrow C^{-\infty}(V, E)$.
3. Show that $\Omega^{\mathrm{top}}(V) = \{f : V^n \to \mathbb{R} : f(Av) = \det(A) f(v)\}$.
4. Show the following,
   (a) $\mathrm{Haar}(V)$ is in canonical isomorphism with $\mathrm{Haar}(W) \otimes \mathrm{Haar}(V/W)$.
   (b) $^*\Omega^{\mathrm{top}}(V) \simeq \Omega^{\mathrm{top}}(W) \otimes \Omega^{\mathrm{top}}(V/W)$.
   (c) $^*\mathrm{Ori}(V) \simeq \mathrm{Ori}(W) \otimes \mathrm{Ori}(V/W)$.
   (d) If both spaces are over a non-archimedean field then $Dist_W(V) \simeq Dist(W)$.
   (e) $\mathrm{Haar}(V)^* = \mathrm{Haar}(V^*)$.
5. Find a distribution $\xi \in \mathrm{Dist}(V \backslash W)$ s.t $\nexists \eta \in \mathrm{Dist}(V)$ with $\eta_{|V \backslash W} = \xi$.
6. *Show that the map $\Phi : G_{i-1}/G_i \to C_c^{\infty}(W, \mathrm{Sym}^i(W)^{\perp})$ defined in class is onto.

Generalized functions - Exercise 6

Solve the following exercises. Questions marked with (*) are optional.

1. Find a space which is Hausdorff, locally isomorphic to $\mathbb{R}^n$ but is not paracompact.
2. For vector bundles E_1,E_2 define the following notions:
   (a) E_1^*.
   (b) $E_1 \oplus E_2$.
   (c) $E_1 \otimes E_2$.
   (d) For an embedding $\varphi : E_1 \to E_2$, define E_2/E_1.
   (e) $\Lambda^k(E_1)$, $\mathrm{Sym}^k(E_1)$.
   (f) In the real/complex case, define $\mathrm{Dens}(E_1)$.
3. Let M be a smooth manifold.
   (a) Show that the three constructions of the tangent space are equivalent.
   (b) Show that one of these (and thus all three) abide the axioms presented in class.
4. Show that $C^\infty(\mathbb{R}^n, \mathbb{R}^k) = \{f : \mathbb{R}^n \to \mathbb{R}^k : f^*(\mu) \in C^\infty(\mathbb{R}^n) \forall \mu \in C^\infty(\mathbb{R}^k)\}$.
5. * Let $\phi : M \to N$ be a surjective smooth map between connected manifolds.
   (a) Show that ϕ is a proper injective immersion $\iff$ it is a closed embedding.
   (b) Show that if ϕ is a proper submersion, then it is a fibration (fiber bundle).
6. Find an example for an immersion which is not injective.

Generalized functions - Exercise 7

Solve the following exercises. Questions marked with (*) are optional.

1. Let $\varphi : M \to N$ be a map of smooth manifolds. For every point $x \in M$ (and its image) there exists $m, n \in \mathbb{N}$, and homeomorphisms $\rho : \mathbb{R}^m \to M$ and $\psi : \mathbb{R}^n \to N$. Show that:
   (a) If φ is an immersion then $\rho \circ \varphi \circ \psi^{-1}$ is injective.
   (b) If φ is an submersion then $\rho \circ \varphi \circ \psi^{-1}$ is surjective.
   (c) If φ is an étale map then $\rho \circ \varphi \circ \psi^{-1}$ is a homeomorphism.
   (d) * Prove (a)-(c) if φ is a map of F-analytic manifolds and the maps are changed accordingly.
   (Hint: use Guillemin and Pollack as reference.)

2. Let X and Y be ℓ-spaces.
   (a) Show that $C_c^\infty(X) \otimes C_c^\infty(Y) \simeq C_c^\infty(X \times Y)$.
   (b) Find an example such that $C_c^\infty(X)^* \otimes C_c^\infty(Y)^* \not\simeq C_c^\infty(X \times Y)^*$.
   (Hint: consider $X = Y = \mathbb{Z}$.)

3. Show that the topology on $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k \times \mathbb{R}^n)$ that was constructed in class is well defined, i.e:
   (a) Given a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ then it induces a homeomorphism $\varphi^* : C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$.
   (b) Given a smooth map $\psi \in C^\infty(\mathbb{R}^n, \mathrm{GL}_k(\mathbb{R}))$ we have that $\psi_* : C_c^\infty(\mathbb{R}^n, \mathbb{R}^k) \to C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$ is a homeomorphism.

4. Let $f \in C(\mathbb{R}^n)^\infty$, show that $f \in C_c^\infty(\mathbb{R}^n)$ if and only if $|f|_D = \sup_{x \in \mathbb{R}^n} |D(f)(x)|$ is finite for every differential operator $D = \sum_{\vec{i}=0}^{\vec{m}} g_{\vec{i}} \frac{\partial^{\vec{i}}}{\partial x_{\vec{i}}}$ where $g_{\vec{i}} \in C^\infty(\mathbb{R}^n)$.

5. * Given a manifold M and a vector bundle E over it show that the two definitions of the topology on $C_c^\infty(M, E)$ are equivalent (one defined via taking a cover of M and trivialization of E and the other through differential operators).

6. * Show that the two definitions for the density bundle of an F-analytic manifold are equivalent.

Generalized functions - Exercise 8

Solve the following exercises. Questions marked with (*) are optional.

1. Let $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ with $f^{(\vec{i})}(\vec{0}) = 0$ for every $|\vec{i}| < k$, and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism such that $\varphi(0) = 0$. Furthermore let $g \in C^\infty(\mathbb{R}^n, \mathbb{C}^\times)$ be a smooth function, and set $\tilde{f}(x) = f \circ \varphi(x) g(x)$. Show that:
   (a) $\left(\frac{\partial^k}{\partial v_1 \cdot \ldots \cdot \partial v_k} \tilde{f}\right)(\vec{0}) = \left(\frac{\partial^k}{\partial (D\varphi_{v_1}) \cdot \ldots \cdot (\partial D\varphi_{v_k})} f\right)(\vec{0}) g(\vec{0})$.
   (b) Part (a) might not be true if $f^{(\vec{i})}(\vec{0}) \neq 0$ for some $|\vec{i}| < k$.
2. Let $\varphi : X \to Y$ be a map between manifolds (either smooth or F-analytic). Show that if φ is proper then $\varphi^* f \in C_c^\infty(X)$ for $f \in C_c^\infty(Y)$.
3. Let $\varphi : X \to Y$. Show that,
   (a) If φ is a submersion, then $\varphi_*(\mu_c^\infty(X)) \subseteq \mu_c^\infty(Y)$.
   (b) $\varphi_*(Dist_c(X)) \subseteq Dist_c(Y)$.
4. Let $\varphi : X \to Y$ be a submersion. Recall we defined $\varphi^* : C^\infty(Y) \to C^\infty(X)$ both by $\varphi^*(f) = f \circ \varphi$ and by first defining $\varphi^* : C^{-\infty}(Y) \to C^{-\infty}(X)$ via the definition for compactly supported smooth measures, and then by restricting $\varphi_{|C^\infty(Y)}$. Show that the two definitions coincide.
5. Show that $\mathbb{R}^\vee \simeq \mathbb{R}$.
6. * Let $\varphi : X \to Y$ be map of manifolds (either smooth or F-analytic), and set,
$$Dist(X)_{\text{prop},\varphi} = \{\xi \in Dist(X) : \varphi_{|\text{supp}(\xi)} \text{ is proper}\}.$$
   Show that the definition for pushing forward $Dist(X)_{\text{prop},\varphi}$ as given in class is well defined (such ρ as was demanded in class exists and the definition is independent of its choice).

Generalized functions - Exercise 9

Solve the following exercises. Questions marked with (*) are optional.

1. Let G be a locally compact abelian group. For an abelian group H and a character $\tau : H \to \mathbb{R}$ define $sh_h(\tau)(x) = \tau(x+h)$. Show that for $\eta \in \mu_c(G)$ and $g \in G$:
   (a) $\mathcal{F}(sh_g(\eta))(\chi) = \chi(g)\mathcal{F}(\eta)(\chi)$ for all $\chi \in G^\vee$.
   (b) $\mathcal{F}(\chi\eta) = sh_\chi(\mathcal{F}(\eta))$ for all $\chi \in G^\vee$.
2. Let F be a non-Archimedean local field, show that $\mathcal{F}(S(V, Haar(V))) \subset S(V^\vee)$.
3. Let V be a finite dimensional topological vector space. Show that $Haar(V^\vee) \simeq^{can} Haar(V)^*$.
   (Hint: use the map that was defined in class.)
4. Using the notations defined in class, show that $\mathcal{F}_1 \circ \mathcal{F}_0(\xi) = \xi^{\text{fl}}$ for all $\xi \in S^*(V)$, where $\langle \xi^{\text{fl}}, f\rangle = \langle \xi, f \circ ()^{-1}\rangle\rangle$.
5. Let $V/\mathbb{R}$ be a 1-dimensional vector space with a positive structure. Show that,
   (a) $V \simeq^{can} |V|$.
   (b) $V^{\alpha+\beta} \simeq^{can} V^\alpha \otimes V^\beta$ where $\alpha, \beta \in \mathbb{Q}^\times$.
6. * Let F be a local field, show that $F^\vee \simeq F$.
7. * Show that every continuous character $\chi : F \to U_1(\mathbb{C})$ where F is a local field is smooth (i.e. in $C^\infty(F)$).
8. $*$* Show that the wavefront set as defined in class is unique.

Generalized functions - Exercise 10

The following exercise are mainly a collection of exercises given in the last few lectures, feel free to solve whichever seems interesting.

1. Let V be a vector space of a local field F, and recall that $\mathbb{P}(V)$ is the projective space over V. Show that

$$\mathbb{P}(V^* \oplus F^*) = \{L \subseteq V : \dim(V/L) = 1\}.$$

2. Show that $f \in C^\infty(V)$ vanishes along v for all $0 \neq v \in V \iff f \in \mathcal{S}(V)$.

3. Let M be a smooth or F-analytic manifold, E a real vector bundle over it and $\xi \in C^{-\infty}(M, E)$. Show that for Hormander's definition of the wavefront set we have the following properties:

   (a) Property 1: $WF(\xi)$ is closed.

   (b) Property 2: the wave front set is conical, if $(x, l) \in WF(\xi)$ then $(x, \alpha l) \in WF(\xi)$ for all $\alpha \in F$.

   (c) Property 3: $p_M(WF(\xi)) = WF(\xi) \cap M = \text{supp}(\xi)$ where $p_M : T^*M \to M$ is the projection.

   (d) Property 4: $\xi \in C^\infty(M, E) \iff WF(\xi) \subseteq M \subset T^*M$.

   (e) Property 5: $WF(f\xi_1 + g\xi_2) \subseteq WF(\xi_1) \cup WF(\xi_2)$ for every $f, g \in C^\infty(M)$ and $\xi_1, \xi_2 \in C^{-\infty}(M, E)$.

   (f) Property 8: For a submersion $\varphi : M \to N$ and $\eta \in C^{-\infty}(N, E')$ we have that $WF(\varphi^*(\eta)) = \varphi^*(WF(\eta))$.
   (Hint: reduce to the case of $\varphi : W \times V \to V$)

   (g) Property 9: For a map $\varphi : M \to N$ and $\xi \in C^{-\infty}(M, E)$ and $\xi \in C^{-\infty}_{\text{prop},\varphi}(M)$ we have that $WF(\varphi_*(\xi)) \subseteq \varphi^*(WF(\xi))$, in several steps:

      i. Where φ is an injective map between vector space, $\varphi : V \hookrightarrow W$.

      ii. Where φ is a closed embedding, $\varphi : M \hookrightarrow N$.

      iii. Where φ is a projection between vector space, $\varphi : V \times W \to V$.

      iv. The general case (hint: use the fact that a map between manifolds can be written as a composition of a closed embedding (to its graph) and a projection).

   (h) Property 0 for affine maps: if $\varphi : V \to V$ is affine (and $M = V$) then $WF(\xi) = WF(\varphi_*(\xi))$.

4. Let F be a non-Archimedean field. Show that $\mathcal{F}(1_{\epsilon\mathcal{O}_F}) = c1_{\frac{1}{\epsilon}\mathcal{O}_F}$, for some $c \in \mathbb{C}$ where 1_X is the indicator function of the set $X \subset V$ and $\mathcal{O}_F$ is the unit ball in F.

5. Let F be a non-Archimedean local field, construct a non-trivial character $\chi : F \to \mathbb{C}^\times$.

6. Calculate the following:

   (a) $WF(|x|)$.

   (b) $WF(|x|^{-\frac{1}{2}})$.

(c) $WF(|x^2+y^2-1|^{-\frac{1}{2}})$.

7. Let V be a vector space and $\Gamma \subset T^*V$ a closed set, and define $D = \{(v,l,\epsilon) \in T^*V \times \mathbb{R}_{>0} : B_\epsilon(v) \times B_\epsilon(l) \cap \Gamma = \varnothing\}$. Show that,

$$C_\Gamma^{-\infty}(V) = \{\xi \in C^{-\infty}(V) : \forall (v,l,\epsilon) \in D, \text{ we have } m^*\mathcal{F}(\rho_{\epsilon,v}\xi)|_{B_\epsilon(l)\times F} \in S(B_\epsilon(l)\times F)\},$$

where $\rho_{\epsilon,v}$ is a cutoff function on $B_\epsilon(v)$.

8. Let V be a vector space and $\Gamma \subset T^*V$ a closed set. Show that $C_c^\infty(V)$ is dense in $C_\Gamma^{-\infty}(V)$ with the topology on $C_\Gamma^{-\infty}(V)$ that was defined in class.

9. Let M and N be manifolds, $\Gamma \subset Y$ and $\varphi : M \to N$ such that $\Gamma \cap S_\varphi \subset Y$, where $S_\varphi = \{(\varphi(x), y) \in T^*N : d_x^*\varphi(y) = 0\}$. Show that $pr_Y^*(\Gamma) \cap S_i \subset X \times Y$ for $i : X \to X \times Y$ via $i(x) = (x, \varphi(x))$.

10. Let M and N be manifolds, show that if $\varphi : M \hookrightarrow N$ is an embedding, then $S_\varphi = CN_M^N$.

11. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and $A = \sum f_i \frac{\partial^i}{\partial x_i}$ a differential operator. Show that:

    (a) $\text{Symb}(\varphi^*A) = \varphi^*\text{Symb}(A)$.

    (b) $\text{Symb}(\varphi_*A) = \varphi_*\text{Symb}(A)$.

12. Let A be a differential operator, M a smooth manifold and $\xi \in Dist(M)$. Show that $WF(A\xi) \subset WF(\xi)$.

13. Let V be a vector space over a local field. Show that $f \in C^\infty(V)$ vanishes asymptotically along $v \in V \iff \exists \rho \in C_c^\infty(V)$ with $\rho(v) \neq 0$ and $\rho m^*(f) \in S(V \times F)$, where $m : V \times F \to V$ is the multiplication map, $m(v,\lambda) = \lambda v$.

14. Prove the following theorem under the assumptions given bellow; let $\varphi : M \to N$ be a map between manifolds, $\Gamma \subset T^*N$ a closed set such that $\Gamma \cap S_\varphi \subset N$. Then $\varphi^* : C^\infty(N) \to C^\infty(M)$ can be extended continuously to $\varphi^* : C_\Gamma^{-\infty}(N) \to C_{\varphi^*(\Gamma)}^{-\infty}(M)$.

    (a) M and N are vector spaces, φ is a linear embedding.

    (b) $\varphi : M \to N$ is a submersion.