Generalized functions - Tirgul

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TA hours: 12:15-13:45, Thursdays.

1 Tirgul 1

Definition 1.1.

1. *Denote by $C^\infty(\mathbb{R})$ the space of smooth functions $f:\mathbb{R}\to\mathbb{R}$ (i.e. having derivatives of any order).*
2. *Denote by $C_c^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ the space of compactly supported function in $C^\infty(\mathbb{R})$ (recall that $\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$).*

Definition 1.2. *We say that $\{f_n\}$ converges to f if*

1. *There is a compact set $K \subset \mathbb{R}$ s.t. $\mathrm{supp}(f) \bigcup\limits_{n\in\mathbb{N}} \mathrm{supp}(f_n) \subseteq K$.*
2. *For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$ (recall that uniform convergence means that the δ chosen can be taken to be independent of x).*

We call the space of continuous functionals $\varphi : C_c^\infty(X) \to \mathbb{R}$, w.r.t. the convergence defined above *distributions*, denote them by $(C_c^\infty(\mathbb{R}))^*$ and write $\langle \varphi, f\rangle$ for $\varphi(f)$. Note it is enough to specify which sequences converge since we consider linear functionals.

Remark 1.3. *For now we have no distinction between generalized functions which we denote by $C^{-\infty}(\mathbb{R})$ and distributions, as there is no difference for $\mathbb{R}$. We will discuss the difference in a later part of the course, when it will be relevant.*

Exercise 1.4. *Prove that $C_c^\infty(\mathbb{R}) \neq \{0\}$.*

Proof. We construct a smooth function with compact support. Assume $a < b$ and consider

$$\eta_{a,b}(x) := e^{-\frac{1}{(x-a)^2}} \cdot e^{-\frac{1}{(x-b)^2}} I_{(a,b)},$$

where $I_{(a,b)}$ is the indicator function of (a,b). Obviously $\mathrm{supp}(\eta_{a,b}) = \overline{(a,b)} = [a,b]$ is compact. It is left to show that η is smooth at a and b (it is smooth at the other points as composition of smooth functions). This is true since $\lim\limits_{x\to a^+} \frac{e^{-\frac{1}{(x-a)^2}}-0}{x-a} = 0 = \lim\limits_{x\to a^-} \frac{0}{x-a}$. The result follows for arbitrary $\eta_{a,b}^{(k)}$ since e^{-x^2} decays faster than any polynomial. ∎

Definition 1.5. *We say that a sequence of functions $\{f_n\}$ weakly converges to f if for every $g(x) \in C_c^\infty(\mathbb{R})$ we have that $\lim\limits_{n\to\infty} \int\limits_{-\infty}^{\infty} g(x) f_n(x)dx = \int\limits_{-\infty}^{\infty} g(x)f(x)dx$.*

Exercise 1.6. *Find a sequence of functions $\{f_n\}$ in $C_c^\infty(\mathbb{R})$ that converges weakly as distributions to the Dirac delta function at zero, δ_0.*

Proof. First note that by definition $\langle \delta_0, g(x)\rangle = \int\limits_{-\infty}^{\infty} g(x)\delta_0(x)dx = g(0)$. Now, consider the sequence of functions $f_n = \frac{\eta_{-\frac{1}{n},\frac{1}{n}}(x)}{I_n}$ where $I_n = \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx$, set $G_n = \max\{g(x)|x \in [-\frac{1}{n}, \frac{1}{n}]\}$ and $g_n = \min\{g(x)|x \in [-\frac{1}{n}, \frac{1}{n}]\}$ and note that for an arbitrary $g(x) \in C_c^\infty(\mathbb{R})$:

$$\int\limits_{-\infty}^{\infty} g(x)f_n(x)dx = \frac{1}{I_n}\int\limits_{-\infty}^{\infty} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx = \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx.$$

Now, note that

$$\frac{g_n}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le \frac{G_n}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \eta_{-\frac{1}{n},\frac{1}{n}}(x)dx.$$

This implies,

$$g_n \le \frac{1}{I_n}\int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\eta_{-\frac{1}{n},\frac{1}{n}}(x)dx \le G_n,$$

yielding the required statement as $\lim\limits_{n\to\infty} G_n = \lim\limits_{n\to\infty} g_n = g(0)$. □

Remark 1.7. *Note that if the sequence f_n converges weakly to f, it need not converge pointwise to f.*

Exercise 1.8. *Find a sequence of functions such that the remark above holds.*

Definition 1.9. *We say a function $f : \mathbb{R}\to\mathbb{R}$ is locally L^1, if $f_{|K} \in L^1(\mathbb{R})$ for every compact $K \subseteq \mathbb{R}$. We denote all such functions by $L^1_{\text{loc}}(\mathbb{R})$.*

Definition 1.10. *Define the derivative of a distribution η by $\langle \eta', f\rangle = -\langle \eta, f'\rangle$.*

Exercise 1.11. *Find a function $f \in L^1_{\text{loc}}(\mathbb{R})$ for which $f' = \delta_0$.*

Proof. Consider the Heaviside step function,

$$H(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x \ge 0. \end{cases}$$

Every compact $K \subseteq \mathbb{R}$ is closed and bounded, so $H(x)$ is locally-L^1. Also, it is smooth and its derivative is 0 in $\mathbb{R}\backslash\{0\}$, but it is not continuous at 0, and thus its derivative is not a function on $\mathbb{R}$. We would like to interpret it as a distribution, indeed using integration by parts:

$$\langle H'(x), g(x)\rangle = \int\limits_{-\infty}^{\infty} g(x)H'(x)dx = g(x)H(x)\Big|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} g'(x)H(x)dx.$$

Recalling that is compactly supported ($g(x) \in C_c^\infty(\mathbb{R})$), and using the fundamental theorem of calculus we see that,

$$g(x)H(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(x)H(x)dx = 0 - \int_{0}^{\infty} g'(x)dx = 0 - (-g(0)) = g(0) = \langle \delta_0, g(x) \rangle.$$

□

Definition 1.12. *For $\xi \in (C_c^\infty(\mathbb{R}))^*$ we say that $\xi_{|U} \equiv 0$ if $\langle \xi, f \rangle = 0$ for all $f \in C_c^\infty(U)$. Additionally we define $\text{supp}(\xi) = \bigcap_{\xi_{|D_\beta^c} \equiv 0} D_\beta$, where D_β are taken to be closed.*

Remark 1.13. *Note that $\text{supp}(\xi)$ is always a closed set.*

Exercise 1.14. *Show the identity axiom for $(C_c^\infty(\mathbb{R}))^*$, i.e. that if there exists $\xi \in (C_c^\infty(\mathbb{R}))^*$ and $\{U_i\}_{i \in I}$ such that $\xi_{|U_i} \equiv 0$ for all $i \in I$, then $\xi_{| \bigcup_{i \in I} U_i}$.*

Proof. Assume we have such cover and such ξ, and take $g \in C_c^\infty(\mathbb{R}) \in \bigcup_{i \in I} U_i$. Use partition of unity to obtain compactly supported functions $f_\alpha : \mathbb{R} \to [0, 1]$ such that for every α we have that $\text{supp}(f_\alpha) \subseteq U_i$ for some $i \in I$, and every $x \in \mathbb{R}$ has an open neighborhood U_x such that in U_x for almost all α we have that $f_\alpha \equiv 0$ and $\sum_{f_{\alpha|U_x} \not\equiv 0} f_\alpha = 1$. Since g is compactly supported, $\text{supp}(g) \subseteq \bigcup_{j=0}^{n} U_{x_j}$. Thus, there are finitely many functions $\{f_k\}$ (finitely many are non-zero in each U_{x_j}) such that $\sum_{k=0}^{m} f_k = 1$ for every $x \in \bigcup_{j=0}^{n} U_{x_j}$. Now,

$$\langle \xi, g \rangle = \langle \xi, \sum_{k=0}^{m} g f_k \rangle = \sum_{k=0}^{m} \langle \xi, g f_k \rangle = 0,$$

where the last equality is true since $\xi_{|U_i} \equiv 0$ for every $i \in I$, and since $\text{supp}(g f_k)$ is compact as there exists j such that $\text{supp}(g f_k) \subseteq \text{supp}(f_k) \subseteq U_j$, and $\text{supp}(g f_k)$ is closed by definition. □

Remark 1.15.

1. *Note that the argument for the previous exercise holds for any paracompact space, in particular for any real manifold.*

2. *Warning! It might not be the case that $f_{|U} \in C_c^\infty(U)$ even if $f \in C_c^\infty(V)$ and $U \subset V$.*

Exercise 1.16. *Prove that,*

$$\{\xi \in (C_c^\infty(\mathbb{R}))^* | \ \text{supp}(\xi) = \{0\}\} = < \{\delta^{(k)}\}_{k=0}^{\infty} >_{\mathbb{R}} .$$

We prove two lemmas which will yield the desired result when combined.

Lemma 1.17. *Let ξ be a distribution supported on $\{0\}$, then there exists $k \in \mathbb{N}$ such that $\xi x^k = 0$.*

Proof. Take a bump function ψ such that $\psi \equiv 1$ in some neighborhood of 0 and $\mathrm{supp}(\psi) \subseteq (-1,1)$, and set $\psi_\epsilon(x) = \psi(\epsilon^{-1}x)$. For every $f \in C_c^\infty(\mathbb{R})$ and $\epsilon > 0$, since $0 \notin \mathrm{supp}(f - f\psi_\epsilon)$, we have that $\langle \xi, f - \psi_\epsilon f\rangle = 0$, implying $\langle \xi, f\rangle = \langle \xi, \psi_\epsilon f\rangle$. Since $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ is a continuous linear map, for every compact $D \subseteq \mathbb{R}$ there exists $k_D \geq 0$ and $C_D > 0$ such that for all $f \in C_c^\infty(D)$,

$$|\langle \xi, f\rangle| \leq C_D \big\|f\big\|_{C^{k_D}} = C_D \sup_{x\in D} \sum_{i=0}^{k_D} |f^{(i)}(x)|.$$

Thus for every $d \in \mathbb{N}$ with $k = k_{[-1,1]}$ and $C = C_{[-1,1]}$,

$$\big|\langle \xi x^d, f\rangle\big| = \big|\langle \xi, x^d f\rangle\big| = \big|\langle \xi, x^d f\psi_\epsilon\rangle\big| \leq C \sup_{x\in(-\epsilon,\epsilon)} \sum_{i=0}^{k} |(x^d f\psi_\epsilon)^{(i)}(x)|.$$

Now, note that since $(f\psi_\epsilon)$ is compactly supported and smooth, we can set $M = \max\limits_{x\in[-1,1], i,j\leq k} |\{f^{(i)}(x)\psi^{(j)}(\epsilon^{-1}x)\}|$. Inspect each summand for $x \in [-\epsilon, \epsilon]$,

$$\begin{aligned}
\Big|(x^d f\psi_\epsilon)^{(i)}(x)\Big| &= \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} \Big|(x^d)^{(i_1)} f^{(i_2)}(x)\psi^{(i_3)}(\epsilon^{-1}x)\epsilon^{-i_3}\Big| \\
&\leq \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} d\cdots(d-i_1+1)|x^{(d-i_1)}|M\epsilon^{-i_3} \\
&\leq \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} d\cdots(d-i_1+1)M\epsilon^{d-i_1-i_3}.
\end{aligned}$$

We can now evaluate the entire expression,

$$C \sup_{x\in(-\epsilon,\epsilon)} \sum_{i=0}^{k} |(x^D f\psi_\epsilon)^{(i)}(x)| \leq C \sum_{i=0}^{k} \sum_{i_1+i_2+i_3=i} \binom{i}{i_1, i_2, i_3} d\cdots(d-i_1+1)M\epsilon^{d-i_1-i_3},$$

but since this holds for every $\epsilon > 0$, we can take $d > k+1$ and obtain that $\big|\langle \xi x^d, f\rangle\big| = 0$ for every $f \in C_c^\infty(\mathbb{R})$. ☐

Lemma 1.18. *If $\xi x^k = 0$, that is $\langle \xi x^k, f\rangle = \langle \xi, x^k f\rangle = 0$ for every $f \in C_c^\infty(\mathbb{R})$ then $\xi = \sum\limits_{i=0}^{k-1} c_i \delta_0^{(i)}$ for some $c_i \in \mathbb{R}$.*

Proof. Our strategy will be to prove the claim for $k = 1$, and then to use induction. Assume $\xi x = 0$. First note that for every $f \in C_c^\infty(\mathbb{R})$ we can write,

$$f(x) = f(0) + x \int_0^1 f'(xt)dt. \tag{1}$$

Now, take a test function $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(0) = 1$, and use Formula 1 as above twice to obtain:

$$f(x) = f(0)\psi(0) + x \int_0^1 f'(xt)dt = f(0)\psi(x) - x\Big(f(0) \int_0^1 \psi'(xt)dt - \int_0^1 f'(xt)dt\Big).$$

Note that each summand is a smooth compactly supported function as the two $f(0)$ constant terms cancel in the second summand. Now, we see that,

$$\begin{aligned}\langle \xi, f\rangle &= \langle \xi, f(0)\psi(x)\rangle + \langle \xi, x\Big(-f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\Big)\rangle\\ &= f(0)\langle \xi, \psi(x)\rangle + \underbrace{\langle \xi x, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle}_{=0}\\ &= f(0)\langle \xi, \psi(x)\rangle.\end{aligned}$$

Since $\langle \xi, \psi(x)\rangle$ is independent of ψ a long as $\psi(0) = 1$ (note we can use Formula 1 to write ψ as a combination of any other test function φ as long as $\varphi(0) = 1$), we can set $c_0 = \langle \xi, \psi\rangle$ and we are done. Now, if $\xi x^{k+1} = 0$, then $(\xi x)x^k = 0$ and thus $\xi x = \sum_{i=0}^{k-1} c_i \delta_0^{(i)}$ by induction. We take $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(0) = 1$ and $\psi^{(i)}(0) = 0$ for $1 \le i \le k-1$, using the same expansion as before,

$$\begin{aligned}\langle \xi, f\rangle &= \langle \xi, f(0)\psi(x)\rangle + \langle \xi, x\Big(-f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\Big)\rangle\\ &= f(0)\langle \xi, \psi(x)\rangle + \langle \xi x, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle\\ &= f(0)\langle \xi, \psi(x)\rangle + \sum_{i=0}^{k-1} c_i\langle \delta_0^{(i)}, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle.\end{aligned}$$

Observe that $\langle \delta_0^{(i)}, -f(0)\int_0^1 \psi'(xt)dt + \int_0^1 f'(xt)dt\rangle = \frac{(-1)^i f^{(i+1)}(0)}{(i+1)!}$, since we can use the Taylor expansion of $\frac{f(x)-\psi(x)f(0)}{x}$ at the origin, and since we chose ψ such that $\psi^{(i)}(0) = 0$ for $1 \le i \le k-1$. Since $\langle \xi, \psi\rangle$ is independent of ψ (considering functions ψ with desired conditions), we are done. ☐

Definition 1.19. *Recall that for two functions the convolution is defined via* $f * g = \int_{-\infty}^{\infty} f(t)g(x-t)dt$. *We define the convolution of* $f \in C_c^\infty(\mathbb{R})$ *with a distribution* ξ *by* $(\xi * f)(x) = \langle \xi, f(x-t)\rangle$.

Exercise 1.20. *Show that for every distribution* ξ *and* $f \in C_c^\infty(\mathbb{R})$ *the convolution* $\xi * f$ *is a smooth function.*

Proof. Notice that since ξ is linear and continuous,

$$\begin{aligned}(\xi * f)'(x) &= \lim_{h\to 0} \frac{\xi * f(x+h) - \xi * f(x)}{h}\\ &= \lim_{h\to 0} \frac{\langle \xi, f(x-t+h)\rangle - \langle \xi, f(x-t)\rangle}{h}\\ &= \langle \xi, \lim_{h\to 0} \frac{f(x-t+h) - f(x-t)}{h}\rangle = \langle \xi, f'(x-t)\rangle = (\xi * f')(x).\end{aligned}$$

Thus we see that $(\xi * f)^{(k)} = \xi * f^{(k)}$, and since $f^{(k)} \in C_c^\infty(\mathbb{R})$ for all $k \in \mathbb{N}$ we are done. ☐

Definition 1.21. *For two compactly supported distributions define* $\langle \xi_1 * \xi_2, f\rangle = \langle \xi_1, (\xi_2 * f(-t))(-x)\rangle$. *For the next exercise we also denote* $\overline{f(x)} = f(-x)$ *and* $L_t(f)(x) = f(t+x)$. *Note: this is very bad notation which should not be used elsewhere, if you have a better idea, you are free to tell me.*

Fact 1.22. *Convolution of functions is commutative and associative.*

Exercise 1.23. *Show the following identities for any compactly supported distributions* ξ_1, ξ_2 *and* ξ_3.

1. $\delta_0 * \xi_1 = \xi_1$.
2. $\delta_0' * \xi_1 = \xi_1'$.
3. $\xi_1 * \xi_2 = \xi_2 * \xi_1$.
4. $\xi_1 * (\xi_2 * \xi_3) = (\xi_1 * \xi_2) * \xi_3$.
5. $(\xi_1 * \xi_2)' = \xi_1 * \xi_2' = \xi_1' * \xi_2$.

Proof.

1. $\langle \delta_0 * \xi, f\rangle = \langle \delta_0, \overline{(\xi * \overline{f})}\rangle = (\xi * \overline{f})(0) = \langle \xi, f(t)\rangle = \langle \xi, f\rangle$.
2. $\langle \delta_0' * \xi, f\rangle = \langle \delta_0', \overline{(\xi * \overline{f})}\rangle = -(\xi * \overline{f})'(0) = -\langle \xi, f'\rangle = \langle \xi', f\rangle$.
3. Take an approximation of identity $\eta_n \in C_c^\infty(\mathbb{R})$, and see that,

$$\begin{aligned}\langle \xi_1 * \xi_2, f\rangle &= \langle (\xi_1 * \delta_0), \overline{\xi_2 * \overline{f}}\rangle = \langle \delta_0, \overline{(\xi_1 * \delta_0) * \xi_2 * \overline{f}}\rangle = \langle \delta_0, \overline{\langle \xi_1 * \delta_0, \overline{L_t(\xi_2 * \overline{f})}\rangle}\rangle \\ &= \langle \delta_0, \overline{\lim_{n\to\infty} \langle \xi_1 * \eta_n, \overline{L_t(\xi_2 * \overline{f})}\rangle}\rangle = \langle \delta_0, \overline{\lim_{n\to\infty} (\xi_1 * \eta_n) * (\xi_2 * \overline{f})}\rangle \\ &= \langle \delta_0, \overline{\lim_{n\to\infty} (\xi_2 * \overline{f}) * (\xi_1 * \eta_n)}\rangle = \langle \delta_0, \overline{\lim_{n\to\infty} \xi_2 * (\xi_1 * \eta_n * \overline{f})}\rangle \\ &= \langle \delta_0, \overline{(\xi_2 * \xi_1 * \delta_0) * \overline{f}}\rangle = \langle \xi_2 * \xi_1, f\rangle.\end{aligned}$$

4. Omitted.
5. Combine the above and see that, $(\xi_1 * \xi_2)' = \delta_0' * (\xi_1 * \xi_2) = (\delta_0' * \xi_1) * \xi_2 = \xi_1' * \xi_2$. For showing the last equality recall that $\langle \xi', f\rangle = -\langle \xi, f'\rangle$, and see that

$$\langle (\xi_1 * \xi_2)', f\rangle = -\langle (\xi_1, \overline{(\xi_2 * \overline{f'})}\rangle = -\langle (\xi_1, \overline{(\xi_2' * (-\overline{f}))}\rangle = \langle \xi_1 * \xi_2', f\rangle.$$

☐

Remark 1.24. *We can now easily construct cutoff functions, that is given a compact* K, *and a neighborhood* $K \subseteq U$ *we can build a function* $\varphi \in C_c^\infty(\mathbb{R})$ *such that* $\varphi_{|K} \equiv 1$, *and* $\varphi \equiv 0$ *outside* U. *To do this take* $2\epsilon = \partial(K, U^c)$ *to be the distance of* K *from* U^c, *and define* K_ϵ *to be the* ϵ *neighborhood of* K. *Then convolve the indicator function of* K_ϵ *with an approximation of identity supported in* $(-\frac{\epsilon}{4}, \frac{\epsilon}{4})$. *This yields a function* $1_{K_\epsilon} * \eta_{\frac{\epsilon}{4}}$ *which equals 1 on* K, *vanishes outside of* U, *and decreases smoothly from* K_ϵ *to* $K_{\frac{3\epsilon}{2}}$.

2 Tirgul 2

Definition 2.1. *A topological vector space V is a vector space over a field F such that addition $+ : V \times V \to V$ and multiplication by a scalar $\cdot : (F, V) \to V$ are continuous functions. Throughout these notes (and in the course) we will also assume V is Hausdorff.*

Definition 2.2. *Let V be a topological vector space over F.*

1. *We say that a set $A \subseteq V$ is convex if for every $a, b \in A$ the linear combination $ta + (1-t)b \in A$ where $t \in [0, 1]$.*
2. *We say that V is locally convex if it has a basis of its topology which consists of convex sets.*
3. *For every open convex set $0 \in C$ in V we set ($x \in V$):*
$$N_C(x) = \inf\{\alpha \in \mathbb{R}_{\geq 0} : \frac{x}{\alpha} \in C\}.$$
4. *We say that a set $W \subseteq V$ is balanced if $\lambda W \subseteq W$ for all $|\lambda| \leq 1$ where $\lambda \in F$.*

Exercise 2.3. *Find a topological vector space which is not locally convex.*

Proof. For $0 < p < 1$ define $\|x\|_p = \sum\limits_{i=0}^{\infty} |x_i|^p$ and consider the space

$$\ell^p(\mathbb{C}) = \{(x_n)_{n\in\mathbb{N}} : x_i \in \mathbb{C}, \|x\|^p < \infty\},$$

with the topology induced by the metric $d(x, y) = \|x - y\|_p$. We claim it is not locally convex. Indeed, if it was locally convex then in any open ball $B_r(0)$ around 0 with radius r we would have an open convex set C_B, which will in turn contain a smaller open δ-ball, denote it by $B_\delta(0)$. The convex hull of $B_\delta(0)$ is then contained in C_B, but taking the following convex combination we see this cannot be true:

$$\left|\frac{1}{n}(\delta, 0, 0, \dots) + \frac{1}{n}(0, \delta, 0, \dots) + \dots + \frac{1}{n}(0, \dots, \underbrace{\delta}_{x_n}, 0, \dots)\right|_p \leq \sum_{i=1}^{n} \left(\frac{\delta}{n}\right)^p = \frac{\delta^p}{n^{p-1}} < r,$$

as this inequality should hold for every n, and $\lim\limits_{n\to\infty} \frac{1}{n^{p-1}} = \infty$ for $0 < p < 1$. □

Exercise 2.4. *Let $0 \in C$ be an open convex set in a topological vector space V.*

1. *Show that $N_C(x) < \infty$ for all $x \in V$.*
2. *Show that if furthermore C is balanced then $N_C(x)$ is a semi-norm.*

Proof.

1. Assume the contrary, there exists $v \in V$ such that $\frac{v}{\alpha} \notin C$ for all $\alpha \in \mathbb{R}$. Thus, $v_n = \frac{v}{n}$ is a sequence in the closed set C^c, but it converges to $0 \notin C^c$, this is a contradiction.
2. Easy to check.

□

Exercise 2.5. *Let $0 \in C$ be an open convex set in a topological vector space V. Find a locally convex topological vector space V such that V has no continuous norm on it.*

Exercise 2.6. *Let V be a locally convex linear topological space. Prove that V is Hausdorff iff $\{0\}$ is a closed set.*

Proof. ($\Rightarrow$:) Assume V is Hausdorff, then for every $a \in V$, there exist open sets $a \in U_1$ and $0 \in U_2$ such that $U_1 \cap U_2 = \varnothing$. In particular $a \in U_1 \subseteq \{0\}^c$.
($\Leftarrow$:) If $\{0\}$ is a closed set, then so is any $\{a\}$ where $a \in V$, as addition is continuous and invertible. It is enough to show that for every such a there exist $a \in U_1$ and $0 \in U_2$ such that $U_1 \cap U_2 = \varnothing$. Consider $\{0\}^c$, it is open and since addition is continuous the set $+^{-1}(\{0\}^c) \subseteq V \times V$ is open. Since V is locally convex, so is $V \times V$ with the product topology, and thus there exist convex open U_1 and U_2 such that $(0, a) \in U_1 \times U_2$ and $U_1 + U_2 \subseteq \{0\}^c$. Now, consider $0 \in -U_1$ and $a \in U_2$. If $(-U_1) \cap U_2 \neq \varnothing$, there exists $t \in (-U_1) \cap U_2$, but this contradicts $0 \notin U_1 + U_2$. □

Exercise 2.7. *Let V be a topological vector space, show that for every neighborhood U of 0 there exists an open balanced set W such that $0 \in W \subseteq U$.*

Exercise 2.8. *Show that every finite dimensional vector space V which is Hausdorff is isomorphic to F^n.*

Proof. Take the standard basis $\{e_i\}_{i=1}^n$ of F^n and choose a basis $\{x_i\}_{i=1}^n$ of V. Set $\varphi : F^n \to V$ by $\varphi(e_i) = v_i$, it is an isomorphism of vector spaces, we would like to show it is also a homeomorphism. First, note that since addition and multiplication by scalar are continuous, so is φ, since we can view it as a composition of two functions $F^n \to (F \times V)^n \to V$, where the first is the injection $(\lambda_1, \dots, \lambda_n) \mapsto ((\lambda_1, x_1), \dots, (\lambda_n, x_n))$, and the second map is multiplication and the addition. To show that φ is open, we do the following. First consider the unit sphere

$$S_F^n = \{x \in F^n : \|x\|_{F^n} = 1\} =\subseteq F^n.$$

It is compact, and since φ is continuous $\varphi(S_F^n)$ is compact, and thus closed as is V Hausdorff. Now, the complement $\varphi(S_F^n)^c$ is open, contains 0, and by an exercise it must contain a balanced neighborhood W of 0. Since φ^{-1} is linear, $\varphi^{-1}(W)$ is a balanced neighborhood of 0 in F^n, and by the construction if $x \in \varphi^{-1}(W)$, we have that $\|x\|_{F^n} < 1$ (why?). Now, for each $1 \leq i \leq n$ let $\ell_i : F^n \to F$ be the projection map to the i-th coordinate. Since $\ell_i \circ \varphi^{-1} : V \to F$ is a bounded linear functional on a neighborhood of 0 as $|\ell_i \circ \varphi^{-1}(x)|_F < 1$ for all $x \in W$, it is continuous (this is not hard, see [6, Theorem 6.21] for details), and thus $\varphi^{-1} = \sum_{i=1}^{n} \ell_i \circ \varphi^{-1} e_i$ is continuous. □

Remark 2.9. *If V is furthermore locally convex, we can finish off the argument by showing that in the image of each open ϵ-ball around 0 under φ there exists around each point an open convex set, since the open balls are a basis for the topology of F^n we are done.*

Exercise 2.10. *Let $W \subseteq V$ be locally convex topological vector spaces, and set V' and W' to be the continuous duals of V and W respectively, and let $()^*$ denote the usual dual.*

1. *Show that the restriction map $V^* \to W^*$ is onto.*
2. *Show that the restriction map $V' \to W'$ is onto.*

Theorem 2.11. *(Hahn-Banach) Let V be a normed vector space, $W \subseteq V$ a closed subset and let $f : W \to \mathbb{R}$ be a linear functional such that $\exists 0 < C \in \mathbb{R}$ such that $|f(x)| < C\|x\|$ for every $x \in W$. Then, there exists a linear functional $\tilde{f} : V \to \mathbb{R}$ extending f such that $\tilde{f}_{|W} = f$ and $|\tilde{f}(x)| < C\|x\|$ for every $x \in V$.*

Exercise 2.12. *Let V be a locally convex topological vector space, and let $f : W \to \mathbb{R}$ be a continuous linear functional, where $W \subseteq V$ is a closed linear subspace of V. Show that f can be extended to V.*

Proof. Recall that the topology of V is generated by open convex sets, each of which corresponds to a semi-norm. f is continuous, and thus it is bounded with respect to some semi-norm $N_C(x)$ where $0 \in C$ is convex and open, we thus have:

$$|f(x)| \leq MN_C(x).$$

Note that this follows since $0 \in f^{-1}(-\epsilon, \epsilon)$ contains an open convex set, and we can take an open convex inside of it. Now, consider the topological vector space $\tilde{V} = V/\ker N_C(x)$, and denote the projection by $p : V \to \tilde{V}$. Since we have a bound on f, if $N_C(w) = 0$ we get that $|f(w)| \leq MN_C(w) = 0$ and thus $\ker N_C \subseteq \ker(f)$, implying that f is defined on $\tilde{V}$, by setting $f(\bar{w}) = f \circ p^{-1}(\bar{w})$. On $\tilde{V}$ the semi-norm $N_C(x)$ is a norm, and thus we can use the Hahn-Banach theorem to extend $f \circ p^{-1}$ to $\tilde{f}$, with the same bound. To finish off the argument, define the extension of f to V to be $\tilde{f} \circ p$.

□

Definition 2.13. *Let V be a topological vector space.*

1. *We say that a sequence $\{v_n\}_{i=1}^{\infty}$ is a Cauchy sequence if for every neighborhood U of 0 there is $n_0 \in \mathbb{N}$ such that $v_n - v_m \in U$ for all $m, n > n_0$.*
2. *We say that $\{v_n\}_{i=1}^{\infty}$ converges to ℓ if for every neighborhood U of ℓ there is $n_0 \in \mathbb{N}$ such that $v_n \in U$ for all $n > n_0$.*
3. *V is called sequentially complete if every Cauchy sequence $\{v_n\}_{i=1}^{\infty}$ converges to some $\ell \in V$.*
4. *V is called complete if for every map $\phi : V \to W$ which maps V homeomorphically into $\phi(V)$, the image $\phi(V)$ is closed in W.*

Exercise 2.14. *Find a topological space which is sequentially complete but is not complete.*

Proof. Taken from [4, Chapter 2 Example 3], this argument holds for any field K with its natural topology. Set $X_d = \mathbb{R}^d$ for $d > \aleph_0$, where $\mathbb{R}$ is equipped with its natural topology, and take $H \subseteq X_d$ to be the subspace of vectors with only countably many non-zero entries. Note that a basis for the topology of

$\mathbb{R}^d$ is comprised of sets $\prod_{i\in d} U_i$ such that each U_i is open in $\mathbb{R}$ and only finitely many $U_i \neq \mathbb{R}$. Now, H is sequentially complete, since if $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then so is $v_n(\alpha)$ (where $\alpha \in d$), and it converges in $\mathbb{R}$. We can thus define v by $v(i) = \lim_{n\to\infty} v_n(i)$, for each $i \in d$. Since each v_n had only countable many non-zero entries, v can only countably many entries which are non-zero and thus $v \in H$. Take a basic open set $W = \prod_{i\in d} W_i$ around v, since only finitely many $W_i \neq \mathbb{R}$, there is $n_0 \in \mathbb{N}$ such that if $n > n_0$ all $v_n \in W$, and thus H is sequentially complete. H is dense in X_d since every basic open set of X_d intersects H as it only has finitely many elements of the product which are different than $\mathbb{R}$, and thus it is not a complete space. □

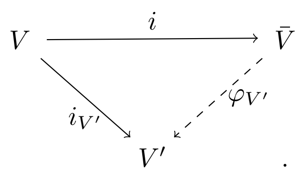
Definition 2.15. *Let $\bar{V}$ be a complete space, and $i : V \to \bar{V}$ be an embedding. We say that $\bar{V}$ is a completion of V if one of the following equivalent definitions holds:*

1. *$i(V) \simeq V$ and $\overline{i(V)} \simeq \bar{V}$.*
2. *For every complete W and a map $f : V \to W$ there exists a unique map $\varphi_W : \bar{V} \to W$ such that $f = \varphi_W \circ i$.*

If $V \simeq \bar{V}$ we say that V is complete.

Exercise 2.16. *Show that these two definitions are indeed equivalent.*

Proof. (2) ⇒ (1): We have $i : V \to \bar{V}$, set $V' = \overline{V} \subseteq \bar{V}$, it is closed in a complete space and thus complete (proof uses Cauchy filters, see Proposition 5.4 of [5]), we then have $i_{V'} : V \to V'$ by restricting the range of i to V'. We thus have a unique map $\varphi_{V'} : \bar{V} \to V'$ such that the following diagram is commutative,



Since we also have the injection $j : V' \to \bar{V}$, we get two maps, $j \circ \varphi_{V'} : \bar{V} \to \bar{V}$ and $\varphi_{V'} \circ j : V' \to V'$, since by the universal property of (2) the only map from a complete space to itself is the identity map, and both V' and $\bar{V}$ are complete we get that $j \circ \varphi_{V'}$ and $\varphi_{V'} \circ j$ are the identity maps, and thus $\bar{V} \simeq V' = \overline{i(V)}$. Now, note that we can also assume that we have a completion w.r.t (1) (this always exists by the next exercise), and thus we have a map $i' : V \to \hat{V}$ such that $\overline{i'(V)} = \hat{V}$ and V is homeomorphic to its image. By the universal property of (2) we then have a continuous $f : \bar{V} \to \hat{V}$ such that $i' = f \circ i$, but then if $U \subseteq V$ is open, so is $i'(U)$, implying that $f^{-1}(i'(U)) = f^{-1} \circ f \circ i(U)$ is open. Since $i(V)$ is dense in $\bar{V}$, f is determined by the image of $i(V)$ in $\hat{V}$, thus $f^{-1} \circ f \circ i(U) = i(U)$ and is open in $\bar{V}$. This finishes the proof.
(1) ⇒ (2): Assume $\overline{i(V)} = \bar{V}$ and i maps V homeomorphically into its image in $\bar{V}$. Assume we have a map $f : V \to W$, then since V is dense in $\bar{V}$ it determines uniquely a map $\varphi_W : \bar{V} \to W$ such that $f = \varphi_W \circ i$. □

Exercise 2.17. *Let V be a topological vector space, show it has a completion.*

Proof. Construction uses either Cauchy filters or Cauchy nets, see [5, Theorem 5.2] for details. □

Definition 2.18. *A topological space (X, τ) is said to be metrizable if there exists a metric which induces the topology τ on X.*

Definition 2.19. *We say that a topological space is S_1 or first countable if every point has a countable basis of open sets.*

Exercise 2.20. *Show that in the category of first-countable vector spaces V is complete if and only if it is sequentially complete.*

Proof. This is essentially a statement about filters, see [5, Prop. 8.2]. □

Definition 2.21. *A Fréchet space is a locally convex, complete and metrizable topological vector space.*

Example 2.22. *Let K be a compact set, and $k, n \in \mathbb{N}_0$. $C^\infty(\mathbb{R}^n)$ and $C_K^\infty(\mathbb{R}^n)$ are Fréchet. $C_K^k(\mathbb{R}^n)$ is a Banach space.*

Exercise 2.23. *Show that for a locally convex, complete topological vector space V the following three conditions are equivalent, thus each implying that V is a Fréchet space.*

1. *V is metrizable.*
2. *V is first countable.*
3. *There is a countable collection of semi-norms $\{n_i\}_{i\in\mathbb{N}}$ that defines the basis for the topology over V.*

Proof. Best to show that metrizable ⇒ first countable ⇒ semi-norms ⇒ metrizable, given as an exercise. □

Exercise 2.24. *Prove that $C^\infty(S^1) \simeq S(\mathbb{Z})$.*

Proof. Both of the spaces are Fréchet, they have the same topology with the homeomorphism given by the Fourier transform. Recall that $\mathcal{F} : C^\infty(S^1) \to S(\mathbb{Z})$ is given by $\mathcal{F}(f) = \{\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x)e^{-inx}dx\}_{n\in\mathbb{Z}}$, and in the other direction by $\mathcal{F}^{-1} : \{c_n\}_{n\in\mathbb{Z}} \mapsto \sum\limits_{n=-\infty}^{\infty} c_n e^{inx}$. Given an ϵ-ball with regards to the semi-norm $n_j(f) = \sup\limits_{x\in S_1} |f^{(j)}(x)|$, its image is given by,

$$\mathcal{F}(B_{n_j,\epsilon}) = \left\{ \{c_n\}_{n\in\mathbb{Z}} : \left| \sum_{n=-\infty}^{\infty} c_n n^i e^{inx} \right| < \epsilon \right\}.$$

Consider the norm m_j of $S(\mathbb{Z})$ given by $m_j(\{c_n\}_{n\in\mathbb{Z}}) = \sup\limits_{n\in\mathbb{Z}} |n^j c_n|$. Note that for $\mathcal{F}(f) = \{c_n\}_{n\in\mathbb{Z}}$, we have for any $m \in \mathbb{Z}$,

$$|m^j c_m| = \left| \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} c_n n^j e^{i(n-m)x} dx \right|$$
$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} c_n n^j e^{i(n-m)x} \right| dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(j)} e^{-imx}| dx$$
$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \sup_{x\in S^1} |f^{(j)}(x)| dx = \sup_{x\in S^1} |f^{(j)}(x)| = n_j(f).$$

This implies that $\sup_{m\in\mathbb{Z}} |m^j c_m| = \|\mathcal{F}(f)\|_{m_j} \leq \|f\|_{n_j}$, meaning that $\mathcal{F}$ is bounded and hence continuous. Alternatively one can view it as $\mathcal{F}(B_{n_i,\epsilon}) \subseteq B_{m_i,\epsilon}$. For the other side, take f in the image of $\mathcal{F}^{-1}$,

$$n_k(f) = \sup_{x\in S^1} |f^{(k)}(x)| = \sup_{x\in S^1} \left| \sum_{n=-\infty}^{\infty} c_n n^k e^{inx} \right| \leq \sum_{n=-\infty}^{\infty} \frac{1}{n^2} |c_n n^{k+2}|$$
$$\leq \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \sup_{m\in\mathbb{Z}} |m^{k+2} c_m| = \frac{2\pi^2}{6} m_{k+2}(\{c_n\}_{n\in\mathbb{Z}}).$$

This implies that $\mathcal{F}^{-1}$ is bounded and thus continuous, which means that $\mathcal{F}$ is open and hence a homeomorphism. □

Exercise 2.25. *Show that the following two bases generate topologies on $C_c^\infty(\mathbb{R})$ which are equivalent,*

1. $U_{k_n,\epsilon_n} = \sum_{n\in\mathbb{N}} \{f \in C^\infty(\mathbb{R}) : \operatorname{supp}(f) \subseteq [-n,n], |f^{(k_n)}| < \epsilon_n\}$.

2. $V_{k_n,\epsilon_n} = \operatorname{conv}_{n\in\mathbb{N}} \{f \in C^\infty(\mathbb{R}) : \operatorname{supp}(f) \subseteq [-n,n], |f^{(k_n)}| < \epsilon_n\}$.

Proof. First note that every two opens balls A_1 and A_2 in $C_c^\infty(\mathbb{R})$, are convex, and that we have that,

$$\frac{A_1 + A_2}{2} \subseteq \operatorname{conv}(A_1, A_2) \subseteq A_1 + A_2 \subseteq \operatorname{conv}(2A_1, 2A_2).$$

Now, for countably many open balls we use the same idea,

$$\sum_{i=1}^{\infty} \frac{A_i}{2^{i+1}} \subseteq \operatorname{conv}_{i\in\mathbb{N}}\{\frac{A_i}{2^i}\} \subseteq \sum_{i=1}^{\infty} \frac{A_i}{2^i} \subseteq \operatorname{conv}_{i\in\mathbb{N}}\{\frac{A_i}{2^{i-1}}\},$$

showing that we can find an open set from (1) in every set of (2) and vice-versa. Note that in both cases we only take finite sums and finite convex combinations. □

Exercise 2.26. *Show that $f_n \in C_c^\infty(\mathbb{R})$ converges to f with respect to the topology defined in the previous question if and only if it converges as was defined in Definition 1.2, i.e,*

1. *There is a compact set $K \subset \mathbb{R}$ s.t.* $\text{supp}(f) \bigcup_{n\in\mathbb{N}} \text{supp}(f_n) \subseteq K$.

2. *For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$.*

Proof. See homework. □

3 Tirgul 3

Definition 3.1. *Let V be a topological vector space, and set $U_{S,\epsilon} = \{\xi \in V^* : \forall f \in S, |\langle \xi, f\rangle| < \epsilon\}$, where V^* is the continuous dual of V.*

1. *We say a set $B \subseteq V$ is bounded if for every open $U \subseteq V$ there exists $\lambda \in \mathbb{R}$ such that $B \subseteq \lambda U$.*

2. *We define the weak topology on V^* by setting the basis of the topology to be $\mathcal{B}_w := \{U_{\epsilon,S} : \epsilon > 0, S \text{ finite}\}$.*

3. *We define the strong topology on V^* by setting the basis of the topology to be $\mathcal{B}_s := \{U_{\epsilon,S} : \epsilon > 0, S \text{ bounded}\}$.*

Remark 3.2. *When the topology on V is given by collection of semi-norms, a set is bounded in V if and only if it is bounded with respect to every semi-norm.*

Theorem 3.3. *(Banach-Steinhaus) Let X be a Fréchet space, Y be a normed vector space, and suppose there is a family F of bounded linear operators $T_\alpha : X \to Y$. If for all $x \in X$ we have that $\sup_{T\in F} \|T(x)\|_Y < \infty$ then,*

$$\sup_{T\in F, \|x\|_k=1} \|T(x)\|_Y < \infty.$$

Proof. This is a version of [6, Theorem 10.18]. □

Exercise 3.4. *The space of distributions $C_c^\infty(\mathbb{R})^*$ is sequentially complete with respect to the weak topology.*

Proof. Take a Cauchy sequence $\{\xi_n\}$ in $C_c^\infty(\mathbb{R})^*$. For every $f \in C_c^\infty(\mathbb{R})$, the limit operator $\xi = \lim_{n\to\infty} \xi_n$ is defined by $\langle \xi, f\rangle = \lim_{n\to\infty} \langle \xi_n, f\rangle$ since $\langle \xi_n, f\rangle$ is a Cauchy sequence of real numbers and thus converges, and is also linear. It is left to show that ξ is continuous, for that, it is enough to show that if $f_n \to f$ then $\langle \xi, f_n\rangle \to \langle \xi, f\rangle$. Let $\{f_n\}_{n=1}^\infty \in C_c^\infty(\mathbb{R})$ be a sequence of functions converging to f in $C_c^\infty(\mathbb{R})$, and set K compact such that $\bigcup_{n=1}^{\infty} \text{supp}(f_n) \cup \text{supp}(f) \subseteq K$. Since each ξ_n is continuous, it is bounded, and thus there exist $C_{K,n} > 0$ and $l_{K,n} > 0$ such that,

$$|\langle \xi_n, f\rangle| \le C_{K,n} \sup_{x\in K} \sum_{i=0}^{l_{K,n}} |f^{(i)}(x)|.$$

Now, since $\langle \xi_n, f\rangle$ is a Cauchy sequence of numbers we get that $\sup_{n\in\mathbb{N}} \langle \xi_n, f\rangle$ is finite. Recalling that $C_K^\infty(\mathbb{R})$ is a Fréchet space, by the Banach-Steinhaus theorem for such spaces, there exist uniform C_K and l_K which bound $|\langle \xi_n, f\rangle|$ for all n, implying that ξ is bounded and continuous. □

Exercise 3.5. *Let $S \subseteq C_c^\infty(\mathbb{R})$ be a bounded set, then $\exists K$ compact such that $S \subseteq C_K^\infty(\mathbb{R})$.*

Exercise 3.6. *Consider the embedding $C_c^\infty(\mathbb{R}) \hookrightarrow C_c^\infty(\mathbb{R})^*$ defined by $f \mapsto \xi_f$. Show that the image of this map is dense in $C_c^\infty(\mathbb{R})^*$ w.r.t both weak and strong topologies.*

Proof. It is enough to show that for every basic open $U \in \mathcal{B}$ there exists $\xi_f \in U$. Start with showing this for compactly supported distributions. Assume that $\text{supp}(\xi) = K$, we need to show that for each basic open set U of 0 we can find ξ_n such that $\xi - \xi_n \in U$. Take an approximation of identity η_n, and define $\xi_n = \xi * \eta_n$. ξ_n are compactly supported functions (by previous exercises), and for each $f \in C_c^\infty(\mathbb{R})$ we have that $\langle \xi - \xi_n, f\rangle \to 0$, implying that for every $U \in \mathcal{B}_w$ we can take n large enough such that $\xi - \xi_n \in U$. Showing this for the strong topology is trickier. By continuity of ξ we obtain for some $k \in \mathbb{N}$,

$$|\langle \xi - \xi_n, f\rangle| = |\langle \xi, f - \overline{\eta_n * \overline{f}}\rangle| \leq C\|f - \overline{\eta_n * \overline{f}}\|_k.$$

Given $U_{S,\epsilon} \in \mathcal{B}_s$, since S is bounded it is contained in $\lambda B(0)_{\|\|_{k+1},1}$ for $\lambda \in \mathbb{R}$. We now show using Lagrange's mean value theorem that $f * \eta_n \to f$ uniformly for $\|\|_k$ (we show this at 0, should be similar at other points).

$$|f^{(k)}(0) - \overline{\eta_n * \overline{f^{(k)}}}| = \left| \int_{-\frac{1}{n}}^{\frac{1}{n}} (f^{(k)}(0) - f^{(k)}(x))\eta_n \right|$$

$$\leq \max_{[-\frac{1}{n},\frac{1}{n}]} |f^{(k)}(x) - f^{(k)}(0)| \leq \max_{[-\frac{1}{n},\frac{1}{n}]} |f^{k+1}(c_x)x| < \frac{\|f\|_{k+1}}{n}.$$

For every $f \in S$ we have that $\|f\|_{k+1} < \lambda$, implying that for all $n \in \mathbb{N}$ we get,

$$\|f - \overline{\eta_n * \overline{f}}\|_k = \sup_{x\in\mathbb{R}} |f^{(k)}(x) - \overline{\eta_n * \overline{f^{(k)}(x)}}| < \frac{\lambda}{n}.$$

Since λ is independent of the function f, we can take $n_0 \in \mathbb{N}$ large enough such that $\xi - \xi_n \in U_{S,\epsilon}$ for $n > n_0$, as required. To finish off the proof, note that compactly supported distributions are dense in distributions by taking elements of the form $\xi_k = \xi I_{[-k,k]}$. We know that $C_c^\infty(\mathbb{R}) = \varinjlim_K C_K^\infty(\mathbb{R})$, and since for every compact K there exists $k \in \mathbb{N}$ large enough such that $K \subseteq [-k, k]$, combining with the fact that $S \subseteq C_K^\infty(\mathbb{R})$ for some K as it is bounded, we get that for n big enough $\xi_n = \xi$ on $S \subseteq C_K^\infty(\mathbb{R})$, and we are done. □

Definition 3.7. *Let $W \subseteq V$ be a closed linear subspace, we define:*

$$V_m(C_c^\infty(\mathbb{R}^n), W) = \{f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^i f}{(\partial x)^i}_{|W} = 0, |i| \leq m\},$$

and

$$F_m((C_c^\infty(\mathbb{R}^n))^*, W) = \{\xi \in (C_c^\infty(\mathbb{R}^n))^* : \xi_{|V_m} = 0\}.$$

Exercise 3.8. *Let $U = \mathbb{R}^n - \mathbb{R}^k$. Show that*

$$\overline{C_c^\infty(U)} = \bigcap_{m=0}^{\infty} V_m = \{f \in C_c^\infty : \frac{\partial^i f}{\partial x_i}_{|\mathbb{R}^k} = 0, |i| \in \mathbb{N}\}.$$

Proof. Given $f \in C_c^\infty(U)$, then $\langle \delta_{\vec{x}}^{\vec{i}}, f\rangle = 0$ if $\vec{x} \in \mathbb{R}^k$ since its support is by definition in U^c, thus $f \in \bigcap_{m=0}^{\infty} V_m$. Since these are continuous operators, they are also zero on the closure, implying that $\overline{C_c^\infty(U)} \subseteq \bigcap_{m=0}^{\infty} V_m$.

For the other direction, given $f \in \bigcap_{m=0}^{\infty} V_m$ use cutoff functions of I_U (which are roughly I_U convolved with approximations of identity, η_n) to construct a sequence of functions which are identically zero in an ϵ-neighborhood of $\mathbb{R}^k$, and thus in $C_c^\infty(U)$, converging to f. □

4 Tirgul 4

4.1 Absolute values

We want to generalize the notion of completion of a field with respect to an absolute value.

Definition 4.1. *A function $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is called an absolute value if for all $x, y \in \mathbb{Q}$:*

1. $|x + y| \leq |x| + |y|$ *(triangle inequality).*
2. $|xy| = |x||y|$.
3. $|x| = 0 \iff x = 0$.

If furthermore $|x + y| \leq \max\{|x|, |y|\}$, $|\cdot|$ is called a non-Archimedean absolute value (and Archimedean otherwise).

Definition 4.2. *We say that two absolute values $|\cdot|_1$ and $|\cdot|_2$ are equivalent if $\exists \alpha \in \mathbb{R}_{\geq 0}$ such that $|\cdot|_1^\alpha = |\cdot|_2$.*

Theorem 4.3. *(Ostrowki's theorem) The only absolute values on $\mathbb{Q}$ up to equivalence are the following:*

1. *The real (Archimedean) absolute value,* $|x|_\infty = \begin{cases} -x, & \text{for } x < 0, \\ x, & \text{for } x \geq 0. \end{cases}$
2. *A p-adic (non-Archimedean) absolute value, if $x = p^n \frac{a}{b}$ and $a, b \in \mathbb{Z}$ are coprime to p, $n \in \mathbb{Z}$,* $|x|_p = \begin{cases} p^{-n}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$
3. *The discrete absolute value,* $|x|_{disc} = \begin{cases} 0, & \text{for } x = 0, \\ 1, & \text{for } x \neq 0. \end{cases}$

Example 4.4. *We have $|1|_7 = |10|_7 = |100|_7 = 1$, and $|49|_7 = |490|_7 = 5^{-2}$.*

Proof. (Ostrowski's theorem) Let $|\cdot|$ be an absolute value, we show it must be one of the above by cases. Assume $|\cdot|$ is non-Archimedean, i.e. $|x + y| \leq$

$\max|x|,|y|$, and set $\mathfrak{a}=\{x\in\mathbb{Z}:|x|<1\}$. This set is non empty as $|0|=0$, and since we assume $|\cdot|$ is non-Archimedean it is an ideal of $\mathbb{Z}$ since,

$$|x+\ldots+x|\leq|x|, \tag{*}$$

and thus if $x\in\mathfrak{a}$, meaning that $|x|<1$, then $xy=\underbrace{x+\ldots+x}_{y\text{ times}}\in\mathfrak{a}$. Consider a prime p. If $|p|=1$ for every prime, we get that $|x|=1$ for every $0\neq x\in\mathbb{Q}$, as $|\frac{1}{p}|=|p|^{-1}$ (check it!), and thus $|\cdot|$ is the discrete absolute value. Thus we can assume $p\in\mathfrak{a}$ (note that for every integer $|m|\leq 1$ by (*)), implying that $p\mathbb{Z}\subseteq\mathfrak{a}\subsetneq\mathbb{Z}$ and consequentially $p\mathbb{Z}=\mathfrak{a}$. Now, if we put $s=-\frac{\log|p|}{\log p}$, we see that $|p|=p^{-s}$. Taking $x=p^n\frac{a}{b}$ where $a,b,n\in\mathbb{Z}$ and a and b are coprime to p we get:

$$|x|=|p^n\frac{a}{b}|=|p^n|\underbrace{|\frac{a}{b}|}_{=1}=|p|^n=p^{-ns}=|x|_p^s,$$

showing $|\cdot|$ is equivalent to $|\cdot|_p$.

Now, assume $|\cdot|$ is an Archimedean absolute value. We must have that $|n|>1$ for $n\in\mathbb{N}$, we do it by induction starting with $n=2$. If $|n|\leq 1$, take $n<x\in\mathbb{N}$ and write it in base n:

$$x=a_0+a_1n+a_2n^2+\ldots+a_rn^r,\ \text{for } 0\leq a_i\leq n-1,\ n^r\leq x.$$

We get that $|a_i|\leq a_i\leq n$, thus

$$|x|\leq\sum_{i=0}^{r}|a_in^i|\leq\sum_{i=0}^{r}n|n|^i=\frac{n(1-|n|^{r+1})}{1-|n|}\leq\frac{n}{1-|n|}.$$

Since this is constant (independent of r), we must have that $|x|\leq 1$, in contradiction to the fact that $|\cdot|$ is Archimedean, as otherwise we could take powers of x and get $x^k>\frac{n}{1-|n|}$ for k big enough. We can thus assume $|n|>1$ for $1<n\in\mathbb{N}$. Note that $r\leq\frac{\log x}{\log n}$, we now have:

$$|x|\leq\sum_{i=0}^{r}|a_i||n|^i\leq(1+\frac{\log x}{\log n})n|n|^{\frac{\log x}{\log n}}.$$

Using these bounds for $x^k<n^{(r+1)k}$:

$$|x|^k\leq k(1+\frac{\log x}{\log n})n|n|^{(k+1)\frac{\log x}{\log n}},$$

implying

$$|x|^{\frac{1}{\log x}}\leq\sum_{i=0}^{r}|a_i||n|^i\leq\sqrt[k]{k(1+\frac{\log x}{\log n})n|n|^{\frac{(k+1)}{k\log n}}}.$$

By taking $k\to\infty$, we get $|x|^{\frac{1}{\log x}}\leq|n|^{\frac{1}{\log n}}$. But,

$$|n|^{\frac{1}{\log n}}=|x^{\frac{\log|n|}{\log x\log n}}|\leq|x|^{\frac{1}{\log x}},$$

thus $|x|^{\frac{1}{\log x}}=|n|^{\frac{1}{\log n}}=e^{\frac{\log|x|}{\log x}}$ is constant, implying that $s=\frac{\log|x|}{\log x}=\frac{\log|n|}{\log n}$ is constant. Now, note that $|x|=x^s$ for every x and get,

$$|x|=x^s=|x|_\infty^s,$$

finishing the proof. □

4.2 p-adic numbers

Definition 4.5. *We define the p-adic numbers* $\mathbb{Q}_p$ *to be the completion of* $\mathbb{Q}$ *with respect to the absolute value* $|\cdot|_p$.

Remark 4.6. *We get a space which is an uncountable field of characteristic* 0, *not algebraically closed, locally compact (every point has a compact neighborhood) and totally disconnected, i.e. every connected component is a point.*

Definition 4.7. *We define the p-adic integers* $\mathbb{Z}_p$ *to be the unit disc in* $\mathbb{Q}_p$, *explicitly* $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

Exercise 4.8. *Show that* $\sum\limits_{n=0}^{\infty} a_n$ *converges* $\iff |a_n|_p \to 0$.

Proof. One direction is the same as in usual real analysis; If the sum converges then the partial sums are a Cauchy sequence thus for every $\epsilon > 0$ there exists n_0 large enough such that for $n_0 < n_1 < n_2 \in \mathbb{N}$,

$$\left|\sum_{i=0}^{n_2} a_i - \sum_{i=0}^{n_1} a_i\right|_p = \left|\sum_{i=n_1+1}^{n_2} a_i\right|_p < \epsilon.$$

in particular take $n_0 < n-1, n \in \mathbb{N}$ and see that $\left|\sum\limits_{i=0}^{n} a_i - \sum\limits_{i=0}^{n-1} a_i\right| = |a_n| < \epsilon$. For the other direction, assume $|a_n|_p \xrightarrow{n\to\infty} 0$, thus for every $\epsilon > 0$ there exists n_0 large enough such that $|a_n|_p \leq p^{-m} < \epsilon$ for $n_0 < n$, but this exactly means that the sum converges as for $n_0 < n_1, n_2 \in \mathbb{N}$:

$$\left|\sum_{i=0}^{n_2} a_i - \sum_{i=0}^{n_1} a_i\right|_p = \left|\sum_{i=n_1+1}^{n_2} a_i\right|_p \leq \max_{n_1+1\leq i\leq n_2} \{|a_i|_p\} \leq p^{-m} < \epsilon.$$

□

Exercise 4.9. *Show that* $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

Proof. Recall that $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{(x_0, x_1, x_2, \ldots) : x_i = x_j \bmod p^j, j < i\}$, with the topology being the weakest topology such that all the projections $p_n : \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ are continuous, and recall that $\mathbb{Z}/p^n\mathbb{Z}$ all have the discrete topology. We define a map $\varphi : \mathbb{Z}_p \to \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ by $\varphi(a) = (\sum\limits_{i=0}^{k} a_i p^i)_{k=0}^{\infty}$ (we saw that every element in $\mathbb{Q}_p$ can be written as an infinite sum, and since $|x| \leq 1$ for $x \in \mathbb{Z}_p$, their expansion starts on the 0-th term). This is indeed a map of rings, first note that $\varphi(0) = 0$ since 0 is divisible infinitely many times by p and thus $0 = 0 \mod p^n$ for every n, and furthermore $\varphi(1)$ is the constant sequence $(1)_{k=0}^{\infty}$, and it is evidently the unit element of the inverse limit (multiplication is done in each coordinate). To see that the addition and multiplication are sent to the correct elements note that each projection $q_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ by $q_n(x) = x \mod p^n$ is a map of rings, since reducing mod p^n commutes with addition and multiplication. Now, assume $\varphi(a) = 0$, thus $a = 0 \mod p^n$ for every $n \in \mathbb{N}_0$, since it means it is zero in each coordinate, but this must imply that $a = 0$ since this is the only element with $|a|_p = 0$. To see φ is onto, take some a in the right hand side, and construct an element $x \in \mathbb{Z}_p$ such that it the limit of the sequence (a_i). Because a was a compatible sequence in the inverse limit, for every $\epsilon > 0$ we can find n_0 such that for $n_0 < n < m$ we have

$$|a_m - a_n|_p = \left| \sum_{i=n+1}^{\infty} a_i p^i \right|_p \leq p^{-n_0} < \epsilon,$$

implying that (a_i) is a Cauchy sequence with respect to $|\cdot|_p$ in $\mathbb{Z}$, and thus a proper element in $\mathbb{Z}_p$. Now, φ is continuous, given a basic open set $U = p_n^{-1}(a_n) = \{(a_i) \in \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} : a_i = a_n \mod p^n\}$, we see that $\varphi^{-1}(U) = \{x \in \mathbb{Z}_p : x - a_n \in p^n\mathbb{Z}_p\} = a_n + p^n\mathbb{Z}_p$, which is open. Evidently so are inverse images of unions and intersections of such U's. To see φ is open, note that for $a \in \mathbb{Z}_p$:

$$\varphi(a + p^n\mathbb{Z}_p) = \left\{(x_i) \in \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} : x_i = a \mod p^n\right\} = p_n^{-1}(a).$$

Thus, φ is a homeomorphism of topological rings. $\square$

Exercise 4.10. *Show that $\mathbb{Z}_p$ is compact.*

Proof. Using the previous exercise, we know that $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, where the latter is a subspace of $\prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$. Since each $\mathbb{Z}/p^n\mathbb{Z}$ is compact (it is finite), so is their product by Tychonoff's theorem, and since the inverse limit is a closed space of the product, it must be compact. To see it is closed note that $\mathbb{Z}_p = \bigcap_{i=1}^{\infty} C_n$ where $C_n = G_n \times \prod_{m \neq n, n+1} \mathbb{Z}/p^m\mathbb{Z}$ and

$$G_m = \{(x_{n+1} + p^{n+1}\mathbb{Z}, x_n + p^n\mathbb{Z}) : x_n = x_{n+1} \mod p^n\},$$

is the graph of the projection from $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. $\square$

4.3 Properties of ℓ-spaces and analysis on ℓ-spaces

Definition 4.11. *An ℓ-space X is a locally compact, totally disconnected space which is Hausdorff. We furthermore say that X is countable at infinity or σ-compact if it is the countable union of compact sets.*

Definition 4.12. *Let X be an ℓ-space. We say that a function is smooth if it is locally constant, we denote all smooth (i.e. locally constant), compactly supported functions $f : X \to \mathbb{C}$ by $C_c^\infty(X)$ or $S(X)$.*

Exercise 4.13. *Let X be an ℓ-space, show it has a basis of clopen sets (i.e. it is zero-dimensional).*

Proof. Taken from [1, 3.1.7]. Assume we have a point $x \in W \subseteq X$, with W open and $K = \overline{W}$ compact and set $\mathcal{P}_x = \{U \subseteq K : U \text{ is clopen and } x \in U\}$ and $P = \bigcap_{V \in \mathcal{P}_x} V$.

Now, we claim that for every closed subset F of K such that $F \cap P = \varnothing$ there exists some $W \in \mathcal{P}_x$ such that $W \cap F = \varnothing$. Otherwise, set $\eta = \{U \cap F : U \in \mathcal{P}_x\}$. By assumption, it is a family of non-empty closed subsets of F, and since F is compact if $\bigcap_{V \in \eta} V = \varnothing$, then there is a finite collection of V_i such that $\bigcap_{i=0}^{n} V_i = \bigcap_{i=0}^{n} U_i \cap F = \varnothing$ (note that this is an equivalent characterization of compactness via closed sets). Since $\mathcal{P}_x$ is closed under finite intersections, $\bigcap_{i=0}^{n} U_i \in \mathcal{P}_x$, but this is a contradiction since we assumed that every set in $\mathcal{P}_x$ intersects F non-trivially. Thus

$$\varnothing \neq \bigcap_{V \in \eta} V = \bigcap_{U \in \mathcal{P}_x} U \cap F = P \cap F,$$

contradicting the assumption that $P \cap F = \varnothing$, so we have a set $V \in \mathcal{P}_x$ such that $V \cap F = \varnothing$.

We now wish to show that $P = \{x\}$. Assume the contrary, i.e. $P \neq \{x\}$. P is disconnected since X is totally disconnected, so there exists non-empty closed $x \in A$ and B such that $A \cup B = P$ and $A \cap B = \varnothing$ which are open in K. Since K is regular, (Hausdorff + locally compact implies regular), there exist open disjoint sets $A \subseteq U$ and $B \subseteq V$ in X, where we have $F = K \backslash (U \cup V)$ closed in K and $P \cap F = \varnothing$. We showed that for such F we can find $W \in \mathcal{P}_x$ such that $F \cap W = \varnothing$. Now, observe that the open set $G = U \cap W$ is also closed in K as,

$$\overline{G} = \overline{U} \cap W \subseteq (K \backslash V) \cap (K \backslash F) = K \backslash (V \cup F) \subseteq U.$$

Therefore $\overline{G} \subseteq U \cap W = G$. Since $x \in G$, we have $G \in \mathcal{P}_x$, but as $G \cap B = \varnothing$, we get that $P = A \cup B$ is not contained in G, which is a contradiction, implying $P = \{x\}$.

Since for every open neighborhood $x \in O$ the set $K \backslash O$ is compact and $x \notin K \backslash O$, it follows from the above claim that O contains some $V \in \mathcal{P}_x$. □

Theorem 4.14. *Let G be an ℓ-group. There exists up to a factor only one left-invariant distribution $\mu_G \in S^*(G)^G$, that is, a distribution such that:*

$$\langle g_0^{-1} \mu_G, f \rangle = \int_G f(g_0 g) d\mu_G(g) = \int_G f(g) d\mu_G(g) = \langle \mu_G, f \rangle$$

for all $f \in S(G)$ and $g_0 \in G$. Furthermore, we can take $\langle \mu_G, f \rangle > 0$ if $f \in S(G)$ is a non-zero non-negative function. This distribution is a measure which is called a (left-invariant) Haar measure on G.

Proof. We shall start by showing uniqueness. Set the right and left actions of G on itself by $\rho(h)(g) = gh^{-1}$ and $\lambda(h)(g) = hg$, and let $\{N_\alpha\}$ be a fundamental system of compact open subgroups of e_G where $\alpha \in I$. This kind of system exists for $\mathbb{Q}_p$ by taking $\{p\mathbb{Z}_p\}$, and by van-Dantzig's theorem for a general ℓ-space. We can suppose that there exists α_0 such that $N_\alpha \subseteq N_{\alpha_0}$ for all $\alpha \in I$, as if not, pick one N_β and consider the system of neighborhoods $\{N_\beta \cap N_\alpha\}_{\alpha \in I}$. Now, define

$$S_\alpha = \{f \in S(G) : \rho(g) f = f, \forall g \in N_\alpha\},$$

and note that if $N_\alpha \subseteq N_\beta$ we have that $S_\beta \subseteq S_\alpha$. Also, observe that $S(G) = \bigcup_{\alpha \in I} S_\alpha$, since for every $f \in S(G)$ we can write $f \sum_{i=0}^{n} c_i I_{g_i K_i}$ with $I_{g_i K_i}$ being indicator functions of $g_i K_i$, and thus f must be invariant w.r.t the right action of $K = \bigcap_{i=0}^{n} K_i \neq \{e_G\}$ as $K_i k = K_i$ for all $k \in K_i$, and K must contains some N_α.

Each space S_α is invariant to the left action of G since if $\rho(h)I_{g_iK_i} = I_{g_iK_i}$ we also have that $\rho(h)\lambda(g)I_{g_iK_i} = \rho(h)I_{gg_iK_i} = I_{gg_iK_i}$ for all $g \in G$. Thus, if we have a direct system of left-invariant functionals $\mu_\alpha \in S_\alpha^*$, where if $S_\alpha \subseteq S_\beta$ then $\mu_{\beta|S_\alpha} = \mu_\alpha$, there exists a functional on $S(G) = \lim_{\to\alpha} S_\alpha = \bigcup_{\alpha\in I} S_\alpha$.

Now, note that each $f \in S_\alpha$ can be written as a finite sum of $I_{g_iN_\alpha}$ for some $g_i \in G$, as by definition $f(gN_\alpha) = f(g)$, for all $g \in G$ and f is compactly supported which means that g_iN_α are a cover for $\operatorname{supp}(f)$. This implies that S_α is generated by left translates of I_{N_α}, and thus μ_G is unique up to a scalar, since by determining its value on $I_{N_{\alpha_0}}$ we determine its value on every I_{N_α} by $I_{N_{\alpha_0}} = \sum_{i=0}^{[N_{\alpha_0}:N_\alpha]} I_{g_iN_\alpha}$ and thus on every S_α. Now, to construct such μ_G we can define for every $f \in S_\alpha$,

$$\langle \mu_\alpha, f\rangle = \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{g_i \in G/N_\alpha} f(g_i),$$

where this sum is finite since $\operatorname{supp}(f)$ is compact. Now, if $f \in S_\beta \subseteq S_\alpha$,

$$\begin{aligned}\langle \mu_\alpha, f\rangle &= \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{g_i \in G/N_\alpha} f(g_i) = \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{h_j \in G/N_\beta} \left(\sum_{g_i \in N_\beta/N_\alpha} f(h_jg_i)\right) \\ &= \frac{1}{[N_{\alpha_0} : N_\alpha]} \sum_{h_j \in G/N_\beta} ([N_\beta : N_\alpha] f(h_j)) = \frac{1}{[N_{\alpha_0} : N_\beta]} \sum_{h_j \in G/N_\beta} f(h_j) = \langle \mu_\beta, f\rangle.\end{aligned}$$

Which implies that the functionals μ_α are compatible. $\{\mu_\alpha\}$ are also left-invariant (we still sum over all the cosets after applying $\lambda(g)$), and positive, and thus establish the existence of a Haar measure. □

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Definition 5.1. *For a topological vector space V we define* $\operatorname{Haar}(V) \subseteq \mu^\infty(V) \subseteq \operatorname{Dist}(V)$ *to be the one dimensional vector space of Haar measures on it, which exist by Haar's theorem.*

Remark 5.2. *We use above the definition of a Haar measure as a Radon measure, that is a positive (i.e, positive on positive functions) functional on the space of compactly supported continuous functions on V. As we will see in the next exercise, one can define this space either using this definition or using the definition utilized to construct the Haar measure for an ℓ-group in the last exercise session.*

Exercise 5.3. *Let V be a finite dimensional linear space (either an ℓ-space or a real vector space). Show that every distribution on V which is translation invariant is a Haar measure, that is $(C_c^\infty(V)^*)^V = \operatorname{Haar}(V)$.*

Proof. (We show this again for an ℓ-space although we proved we have a Haar measure for an ℓ-group.) If V is an ℓ-space, every $f \in C_c^\infty(V)$ can be written as $f = \sum_{i=0}^{n} c_i I_{v_i+K_i}$ where K_i are compact open subgroups of V. Furthermore, the function f is invariant with respect to $K = \bigcap_{i=0}^{n} K_i$, and thus can be written as a finite sum of translations of K; $f = \sum_{i=0}^{m} c'_i I_{v_i+K}$. Now, if $\int_V f d\mu = 0$, we must have that $\sum_{i=0}^{m} c'_i = 0$ since $\mu(I_K) \neq 0$. Since every $\xi \in (C_c^\infty(V)^*)^V$ is translation-invariant we get that,

$$\langle \xi, f\rangle = \langle \xi, \sum_{i=0}^{m} c'_i I_{v_i+K}\rangle = \langle \xi, \sum_{i=0}^{m} c'_i I_K\rangle = \sum_{i=0}^{m} c'_i \langle \xi, I_K\rangle = 0.$$

We now claim that for $\xi \in (C_c^\infty(V)^*)^V$, if $\langle \xi, f\rangle = 0$ where $\int_V f d\mu \neq 0$, then $\langle \xi, g\rangle = 0$ for all $g \in C_c^\infty(V)$. Indeed, if $\langle \xi, f\rangle = 0$, as we saw before this means that $\sum_{i=0}^{m} c'_i \langle \xi, I_K\rangle = 0$. Since the integral is non-zero, the sum of the coefficients is non-zero and thus $\langle \xi, I_K\rangle = 0$. We get that for every indicator function $I_{K'}$ where $K' \subset V$ is a compact open subgroup the index $[K : K \cap K']$ is finite, implying that,

$$\langle \xi, I_{K'}\rangle = \langle \xi, \sum_{j=0}^{[K':K\cap K']} I_{v_j+K\cap K'}\rangle = [K' : K\cap K']\langle \xi, I_{K\cap K'}\rangle = \frac{[K' : K \cap K']}{[K : K \cap K']}\langle \xi, I_K\rangle = 0.$$

Since this holds for every indicator of a compact open subgroup, this means that $\xi = 0$. To finish off this part of the proof pick some open compact open subgroup K, and note that the map $(C_c^\infty(V)^*)^V \to \mathbb{R}$ by $\xi \mapsto \langle \xi, I_K\rangle$ is an isomorphism since this is a map of vector spaces which is onto (there are non-zero distributions since the space of Haar measures is inside), and we just showed injectivity.

A different way to see this is that the space of integral functions whose integral vanishes, call it W, is of codimension one in $C_c^\infty(V)$, and thus we have that the space we are interested in is actually functionals on the one dimensional space $C_c^\infty(V)/W$.

Now, assume that $V = \mathbb{R}^n$ and ξ some invariant distribution. First note that using continuity of ξ, for every directional derivative,

$$\langle \xi, \frac{\partial f}{\partial \vec{v}}\rangle = \langle \xi, \lim_{h\to 0} \frac{f(x) - f(x - h\vec{v})}{h}\rangle = \lim_{h\to 0} \frac{\langle \xi, f\rangle - \langle L_{h\vec{v}}(\xi), f\rangle}{h},$$

where $L_{h\vec{v}}(\xi)$ denotes translation of ξ by $h\vec{v}$, and thus

$$\langle \xi, \frac{\partial f}{\partial \vec{v}}\rangle = \lim_{h\to 0} \frac{\langle \xi, f\rangle - \langle \xi, f\rangle}{h} = 0.$$

Now, assume that $\langle \xi, f\rangle = 0$ where $\int_V f d\mu \neq 0$. Given any $g \in C_c^\infty(\mathbb{R}^n)$, assume that $\int_V (f - g)d\mu = 0$ (otherwise normalize g). We claim that $f - g = \sum_{i=0}^{n} \frac{\partial f_i}{\partial x_i}$ for some $f_i \in C_c^\infty(\mathbb{R}^n)$ which means that $\langle \xi, f - g\rangle = 0$ and thus $\langle \xi, g\rangle =$

$\langle \xi, g + (f-g) \rangle = \langle \xi, f \rangle = 0$. Indeed, if $n = 1$, take $F(x) = \int_{-\infty}^{x} (f-g)dx$, and this is the fundamental theorem of calculus (note that F is compactly supported since the integral of $f - g$ is zero).

For bigger n, set $h = f - g$ and continue by induction; Set $\alpha(x_n) = \int_{\mathbb{R}^{n-1}} h dx_1 \dots dx_{n-1}$ and $H = h - \alpha(x_n)\Psi(x_1, \dots, x_{n-1})$, where Ψ is a bump function, that is, it is compactly supported and $\int_{\mathbb{R}^{n-1}} \Psi dx_1 \dots dx_{n-1}$=1 and $\alpha(x_n)$ is compactly supported since h is compactly supported. Now, note that for every $x_n \in \mathbb{R}$ we have that, $\int_{-\infty}^{\infty} H dx_1 \dots dx_{n-1} = 0$. Thus by the induction hypothesis, where x_n is fixed, $H = \sum_{i=0}^{n-1} \frac{\partial H_i}{\partial x_i}$ for some $H_i \in C_c^\infty(\mathbb{R}^n)$. Note that H_i are indeed smooth in all variables, including x_n, since they vary smoothly when x_n varies. Now, $h = H + \alpha\Psi$, but we see that $\int_{-\infty}^{\infty} \alpha(x_n) = 0$, and thus $\alpha = \frac{\partial \beta}{\partial x_n}$ for some smooth, compactly supported β Since Ψ is independent of x_n, this implies that $h = \sum_{i=0}^{n-1} \frac{\partial H_i}{\partial x_i} + \frac{\partial \beta \Psi}{\partial x_n}$, and we're done by the same reasoning as in the first part. □

5.1 The exterior algebra

Let V be a finite dimensional vector space, we define the exterior algebra as $\Lambda(V) = \bigoplus_{i=0}^{\dim V} \Lambda^i(V)$, where $\Lambda^k(V) = \bigotimes_{j=0}^{k} V/J$ and J is the ideal generated in $\bigotimes_{j=0}^{k} V$ by the set $\{v_1 \otimes \dots \otimes v_n : v_i = v_j \text{ for some } i \neq j\}$. Note that this implies that the elements of the exterior algebra are anti-symmetric, and that $\Lambda^k(V) = 0$ if $k > \dim V$, since after choosing a basis to V and decomposing an element in $\Lambda^k(V)$ to basic tensors, there must be a basis element which appears twice.

Definition 5.4. *Let V be a finite dimensional vector space of dimension n over F.*

1. *We set $\Omega^k(V) = \Lambda^k(V^*)$.*
2. *For a 1-dimensional space V we define $V^{**} \supseteq |V| = \{f : V^* \to \mathbb{R} : \forall \alpha \in F, f(\alpha v) = |\alpha| f(v)\}$.*
3. *We define the densities of V as $\mathrm{Dens}(V) = \{f : V^n \to \mathbb{R} : f(Av) = |\det(A)| f(v)\}$.*

We can show that $\Omega^n(V) = \{f : V \to \mathbb{R} : f(Av) = \det(A) f(v)\}$, and that this space is one dimensional. We can thus also define $\mathrm{Dens}(V) = |\Omega^{\mathrm{top}}(V)|$.

Exercise 5.5. *Show that* $\mathrm{Dens}(V) \simeq^{can} \mathrm{Haar}(V)$.

Proof. A Haar measure can be viewed both as a functional on compactly supported, coninuous functions and as a function on Borel sets. The absolute value of the determinant $|\det| : V^n \to \mathbb{R}$ is an element of the one dimensional space

$|\Omega^n(V)|$ (recall that for finite dimensional spaces $V \simeq^{can} V^{**}$). We have a canonical isomorphism by choosing a basis $\{e_i\}_{i=1}^n$ for V, and bijecting between the element $\varphi \in |\Omega^n(V)|$ such that $\varphi(e_1, \ldots, e_n) = 1$ with the Haar measure normalized such that it has the value 1 on the parallelogram spanned by the vectors $\{e_i\}_{i=1}^n$. This is independent of choice of basis since given a different basis both elements would be multiplied by the same factor of $|\det(M)|$, where M is the change of basis matrix between these two bases. □

Definition 5.6. *For a topological vector space V which is an ℓ-space we define the space of smooth measures as $S(V, \mathrm{Haar}(V)) \simeq S(V) \otimes \mathrm{Haar}(V)$.*

Remark 5.7. *Note that for such V as in the definition above we have that the smooth measures $S(V, \mathrm{Haar}(V))$ are the compactly supported locally constant functions with values in $\mathrm{Haar}(V)$.*

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Definition 6.1. *We say that a topological space X is paracompact if for every open cover $\{U_\alpha\}_{\alpha \in I}$ of X and point $x \in X$ there is a neighborhood $x \in V$ and a refinement $\{U_\beta\}_{\beta \in J}$ such that V intersects only finitely many sets of $\{U_\beta\}_{\beta \in J}$.*

Definition 6.2. *A topological manifold M is a topological space which is locally homeomorphic to $\mathbb{R}^n$, and is furthermore paracompact and Hausdorff.*

Exercise 6.3. *Find a space X which is locally homeomorphic to $\mathbb{R}^n$ at every point and is paracompact but is not Hausdorff.*

Proof. Consider the space obtained by gluing two copies of $\mathbb{R}$ along an open set, say $(0, \infty)$. At each point we can find a neighborhood small enough which is homeomorphic to $\mathbb{R}$, note that this also works for the points 0_1 and 0_2 arising from the points zero in each copy of $\mathbb{R}$, since an open set around each can be taken only in one copy of $\mathbb{R}$ (recall that the quotient topology is defined to be the weakest such that the quotient map is continuous). This space is also paracompact; take a small neighborhood with a compact closure, we can refine any open cover such that its interior meets only finitely many sets. This space is not Hausdorff since the two points 0_1 and 0_2, cannot be segregated by disjoint open sets, every two such sets must intersect in some interval as $(-\epsilon, 0_1) = (-\epsilon, 0_2)$ for every $\epsilon \in \mathbb{R}_{>0}$. □

Definition 6.4. *An analytic F-manifold is a space M which is locally isomorphic to $\mathcal{O}_F^n$ together with a sheaf of functions*

$$An(U) = \{f : U \to F : \forall x \in U, \exists r > 0 \ s.t. \ f_{|B_r(x)}(y) = \sum_{\vec{k} \in \mathbb{N}^n} a_{\vec{k}} (x - y)^{\vec{k}}\},$$

where $B_r(x)$ is the ball of radius r around x, and $\vec{k}$ is a multi index, thus $(x - y)^{\vec{k}} = \prod_{i=0}^{n} (x_i - y_i)^{k_i}$.

Definition 6.5. *Let M be a smooth manifold or a p-adic analytic manifold. A real vector bundle over M is a tuple (E, p) where E is a topological space and $p : E \to M$ is a continuous surjection such that:*

1. *For every $x \in M$ we have that $p^{-1}(x) = V_x$ is a finite dimensional real vector space.*
2. *For every $x \in M$ there exists an open $x \in U$ and a trivialization $\varphi_U : V_x \times U \to p^{-1}(U)$ where φ_U is a homeomorphism and $p \circ \varphi_U(v, x) = x$ for all $v \in V_x$.*
3. *The maps $v \mapsto \varphi_U(v, x)$ are linear isomorphisms.*

If $E \simeq V \times M$ we say (E, p) is a trivial bundle over M.

Exercise 6.6. *Given a manifold M a vector bundle (E, p) with fiber of constant dimension m over it, and a functor $F : \mathrm{Vect}^m \to \mathrm{Vect}^n$, construct a vector bundle $(F(E), q)$ over M as discussed in class.*

Proof. First, take a cover $\{U_\alpha\}$ which is a local trivialization of E (that is, $p^{-1}(U_\alpha) \simeq V \times U_\alpha$). Define the total space $F(E)$ over each U_α by $F(V) \times U_\alpha$, where the surjection q will be projecting to M, and glue every two pieces $q^{-1}(U_\alpha)$ and $q^{-1}(U_\beta)$ by setting $(v, x) \sim (g_{\alpha,\beta}(v), x)$ for every $x \in U_\alpha \cap U_\beta$ and $v \in V$, where $g_{\alpha,\beta} = F(\varphi_{U_\beta}^{-1}\varphi_{U_\alpha})$. Finally, note that for any two elements of the cover $g_{\alpha,\beta}^{-1} = g_{\beta,\alpha}$, and in order for our construction to be well defined we need to show the cocycle condition , namely that $g_{\beta,\gamma}g_{\alpha,\beta} = g_{\alpha,\gamma}$ when restricted to triple overlaps. This holds since

$$g_{\beta,\gamma}g_{\alpha,\beta} = F(\varphi_{U_\gamma}^{-1}\varphi_{U_\beta})F(\varphi_{U_\beta}^{-1}\varphi_{U_\alpha}) = F(\varphi_{U_\gamma}^{-1}\varphi_{U_\alpha}) = g_{\alpha,\gamma}.$$

Note that if we want $F(E)$ to have a smooth structure we need to demand that F preserves smooth maps. □

We can utilize Exercise 6.6 to define a density bundle over a real manifold:

Definition 6.7. *Let M be a smooth manifold, we can define its density bundle by $D_M = |\Omega^{top}(TM)|$, that is the density bundle of the tangent bundle.*

Definition 6.8. *Let X be an F analytic manifold, we define its density bundle by $D_X = |\Omega^{top}(X)|$.*

Exercise 6.9. *Let M be a smooth n-dimensional Riemannian manifold, that is a smooth real manifold with an inner product on tangent spaces*

$$<,>_p : T_pM \times T_pM \to \mathbb{R}$$

which varies smoothly. Construct explicitly a density over M, that is a smooth section of the density bundle over M. The density should respect coordinate changes, and be the standard density when M is a linear space with the standard inner product.

Proof. Consider the Gram matrix $G_{M,p}(e_1, \ldots, e_n)_{i,j} =< e_i, e_j >_p= (E^T E)_{i,j}$ for $e_i \in T_pM$. This matrix is positive semidefinite, so we can define $vol_{M,p} : (T_pM)^n \to \mathbb{R}$ by

$$vol_{M,p}(e_1, \ldots, e_n) = \sqrt{\det(G_{M,p}(\vec{e}))}.$$

For every $p \in M$ we have that $vol_{M,p} \in |\Omega^{\mathrm{top}}(T_pM)|$ since given $A \in \mathrm{GL}(T_pM)$:

$$vol_{M,p}(A\vec{e}) = \sqrt{\det(G_{M,p}(A\vec{e}))} = \sqrt{\det((AE)^T AE)} = |\det(A)|vol_{M,p}(\vec{e}).$$

This implies that vol_M is a section of D_M, and since we defined it using global constructions (i.e. the tangent bundle), it is independent of coordinates.

Now, to show that $vol_M \in C^\infty(M, D_M)$, for $p \in M$ take an open $p \in U \simeq \mathbb{R}^n$ such that U trivializes D_M. Since the inner product changes smoothly over M and $\det(G_{M,p}) \not\equiv 0$ for every $p \in M$, we have that $vol_{M|_U} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a smooth function as composition of smooth functions and we are done. □

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7.1 Constant sheaves and locally constant sheaves

Definition 7.1. *Let V be a finite dimensional vector space and X a topological space.*

1. *We define the constant sheaf $\underline{V}_X$ to be the sheafification of the constant presheaf, which assigns to every open set in X the vector space V.*
2. *We say that a sheaf $\mathcal{F}$ over X is locally constant if for every $x \in X$ there exists an open $x \in U_x$ and a finite dimensional vector space V_x such that $\mathcal{F}_{|U_x} \simeq \underline{V_x}_{U_x}$.*

Exercise 7.2. *Let V be a finite dimensional vector space and X a topological space.*

1. *Show that $\underline{V}_X(U) = C^\infty(U, V)$.*
2. *Show that if X is a σ-compact ℓ-space then every locally constant sheaf $\mathcal{F}$ where $\mathcal{F}_x \simeq \mathcal{F}_y$ for all $x, y \in X$ is isomorphic to the constant sheaf.*

Proof.

1. First note that $\mathcal{C}_V(U) = C^\infty(U, V)$ is a sheaf as we can glue locally constant functions, and they are determined by their values on a cover of X. Now, note that there is a map of presheaves from the constant presheaf to $\mathcal{C}_V$ by for each U sending $v \in V$ to the constant function with value v on U, which is the identity map on the stalks. Finally, the universal property of the sheafification of $\underline{V}_X$ holds for $\mathcal{C}_V$ since given a sheaf $\mathcal{G}$ and map of presheaves $p : \underline{V}_X \to \mathcal{G}$ it factors through $\mathcal{C}_V$ by sending the constant function with value $v \in \mathcal{C}_V(U)$ to $p(v)$, thus $\mathcal{C}_V \simeq \underline{V}_X$.
2. Take a cover $\{U_\alpha\}$ of X such that $\mathcal{F}_{|U_\alpha} \simeq \underline{V}_{U_\alpha}$. Since X is σ-compact, we can write X as an ascending union of compacts, that is $X = \bigcup_{n=1}^{\infty} K_n$ with $K_n \subset K_{n+1}$. Now, for each K_n we can refine the sets $\{U_\alpha\}$ such that $K_n \cap U_\alpha \neq \varnothing$ to a cover $\{V_\beta\}$ of K_n whose elements are mutually disjoint. Continuing with this process we get $\{V_j\}_{j=0}^\infty$, a countable and mutually disjoint cover of X, where $\mathcal{F}_{|V_i} \simeq \underline{V}_{V_i}$, but gluing the sheaves back together, we must have that $\mathcal{F} \simeq \underline{V}_X$.

□

Remark 7.3. *Note that in the second part of the previous exercise we used the disjointness of the elements of the cover since otherwise we would need to ensure that the cocycle conditions hold on triple overlaps $V_i \cap V_j \cap V_k$. For a precise reference see [7, Exercise 2.7D].*

Remark 7.4. *To make the discussion less cumbersome, for the next two exercises we discuss sheaves of sets.*

Exercise 7.5. *Show that the definition of a Leray sheaf is equivalent to the Grothendieck definition of a sheaf.*

Proof. We will sketch an equivalence of categories between the Leray definition of a sheaf and the Grothendieck one. We start by defining functors in two directions.

Assume we are given a Leray sheaf, that is an espace etale with a projection (E, p) such that for every $e \in E$ there exists a neighborhood $e \in U_e \subseteq E$ such that $p_{|U_e}$ is a homeomorphism to its image. We define a presheaf by,

$$\mathcal{F}(U) = \{f : U \to p^{-1}(U) : f \text{ cts}, \, p \circ f = \mathrm{Id}_U\}$$

with the obvious restriction maps. This is actually a sheaf; because the sections are functions we can glue compatible sections, and the identity axiom holds.

For the other direction, given a Grothendieck sheaf, we form its espace etale by taking $E = \coprod_{x \in X} \{x\} \times \mathcal{F}_x$, with the projection being $p(x, v) = x$. We furthermore endow E with the topology generated by the basis $U_{s,V} = \{(x, (s)_x) : x \in V\}$ where $V \subseteq X$ is open and $s \in \mathcal{F}(V)$. For each $(x, v) = e \in E$ take a basic open set U_{s,V_x}, where $x \in V_x$ and $s \in \mathcal{F}(V_x)$ such that $(s)_x = v$, then p is a homeomorphism from U_{s,V_x} onto $V_x \subseteq X$. Note that p is also continuous since for an open $V \subseteq X$, we have that $p^{-1}(V) = \coprod_{x \in V} \{x\} \times \mathcal{F}_x = \bigcup_{W \subseteq V, \text{open}} \bigcup_{s \in \mathcal{F}(W)} U_{s,W}$, which is open.

We sketch the equivalence of categories. Starting with a Grothendieck sheaf $\mathcal{F}$, we construct a Leray sheaf $(E_{\mathcal{F}}, p)$ and obtain a Grothendieck sheaf $\mathcal{G}_{E_{\mathcal{F}}}$. For an open $U \subseteq X$ the sections are,

$$\mathcal{G}_{E_{\mathcal{F}}}(U) = \{f : U \to \coprod_{x \in U} \{x\} \times \mathcal{F}_x : f \text{ cts}, \, p \circ f = \mathrm{Id}_U\}.$$

Since the basis for the topology of $E_{\mathcal{F}}$ was sets $U_{s,V}$, where $V \subset X$ is open and $s \in \mathcal{F}(V)$, for each s we have a continuous function $f_s \in \mathcal{G}_{E_{\mathcal{F}}}(U)$ where $f_s(x) = (x, (s)_x)$, so $\mathcal{F}(U) \subseteq \mathcal{G}_{E_{\mathcal{F}}}(U)$. Conversely, if we have a continuous section $f : U \to \coprod_{x \in U} \{x\} \times \mathcal{F}_x$, for each $(x, (s)_x)$ in the image, we can consider its germ given by a representative (s, W_s), where $x \in W_s \subseteq U$ is open. By continuity of f, we must have an open $x \in V_s \subseteq f^{-1}(U_{s,W_s})$ with $f(V_s) = U_{s,V_s}$. Since we started with a sheaf $\mathcal{F}$, using the gluing axiom for $s_\alpha \in \mathcal{F}(V_{s_\alpha})$ there exists a section $s \in \mathcal{F}(U)$ corresponding to f (note that they agree on overlaps, since these are values of the function f). This shows that $\mathcal{G}_{E_{\mathcal{F}}} \simeq \mathcal{F}$.

For the other direction, assume we start with a Leray sheaf (E, p), and consider $E_{\mathcal{F}_E} = \coprod_{x \in X} \{x\} \times (\mathcal{F}_E)_x$, where here $(\mathcal{F}_E)_x = \{f : \{x\} \to p^{-1}(x)\}$. We construct a homeomorphism $\psi : E_{\mathcal{F}_E} \to E$ by $\psi(x, (f)_x) = f(x)$. This map is surjective since for every $e \in E$ we have that $\psi(p(e), f_e) = e$ where $f_e \in (\mathcal{F}_E)_{p(e)}$ and $f_e(p(e)) = e$. Injectivity is clear since if $f_1(x) = f_2(y)$, then $x = p(f_1(x)) = p(f_2(y)) = y$ and f_1, f_2 are functions from a singleton.

The map ψ is continuous since given an open $W \subseteq E$, consider $(x, (f)_x) \in \psi^{-1}(W) = \{(x, (f)_x) : f(x) \in W\}$. Since $(f)_x \in (\mathcal{F}_E)_x$, by considering its germ there exists an open subset $V \subseteq p(W)$ (note that $p(W)$ is open) such that $f \in \mathcal{F}_E(V)$, that is $f : V \to p^{-1}(V) \subseteq W$ and is continuous. By the definition of the topology on $E_{\mathcal{F}_E}$, we have that $(x, (f)_x) \in U_{f,V} \subseteq \psi^{-1}(W)$.

To show ψ is an open map, take a basic open $U_{f,W} \subseteq E_{\mathcal{F}_E}$, where $W \subseteq X$ is open and $f : W \to p^{-1}(W)$ is a continuous section. If $e \in \psi(U_{f,W}) = f(W)$, then we have an open $e \in U_e \subseteq E$ such that $p_{|U_e}$ is a homeomorphism. In particular, $p(U_e) \cap W$ is open and since $f_{|p(U_e)\cap W} : p(U_e) \cap W \to p^{-1}(p(U_e)) \cap p^{-1}(W)$ is a continuous section, it must also be a homeomorphism, thus $e \in f(p(U_e) \cap W) \subseteq f(W)$ is open. This finishes the proof.

Remark 7.6. *Note that if we apply the procedure depicted above on a presheaf we end up with a sheaf. This is exactly the sheafification functor from presheaves to sheaves.*

□

Exercise 7.7. *Show that covering spaces correspond to locally constant sheaves, and that a covering space is trivial exactly when it corresponds to a constant sheaf. Give an example for a locally constant sheaf arising from a covering space which is not constant.*

Proof. Assume we are given a covering space (E, p), view it as an espace etale and reconstruct the corresponding Grothendieck sheaf $\mathcal{F}_E$. If X is our base space, we have a cover $\{U_\alpha\}$ of X such that $p^{-1}(U_\alpha) \simeq U_\alpha \times D$ and D is discrete. Thus $\mathcal{F}_{E|U_\alpha}(V) = \{f : V \to V \times D : f \text{ cts}, p \circ f = \mathrm{Id}_U\}$, which are exactly the locally constant functions since D is discrete, implying that $\mathcal{F}_{E|U_\alpha}$ are locally constant sheaves.

Conversely, given a locally constant sheaf with stalk D, assemble its espace etale $(E_\mathcal{F}, p)$. The espace etale of the constant sheaf $\underline{D}_U$ corresponds to $U \times D$ with the product topology, since the stalk at every point is D, and the open sets are $U_{s,V} = V \times \{s\}$. This implies that given a cover $\{U_\alpha\}$ of X such that $\mathcal{F}_{|U_\alpha}$ is isomorphic to the constant sheaf, we will have that $p^{-1}(U_\alpha) \simeq U \times D$, as this is the espace etale of the constant sheaf $\underline{D}_{U_\alpha}$, showing that $(E_\mathcal{F}, p)$ is a covering space.

It can be useful to note that while for a locally constant sheaf the stalks at all points can be isomorphic, if the sheaf is not isomorphic to a constant sheaf the topology of the espace etale will be different than the product topology, as we will see in the example.

Finally, note that we get that a non-trivial covering space cannot gives rise to a constant sheaf by using the equivalence of categories from Exercise 7.5, as otherwise, apply the functors forth and back, and get that the covering space you started with was isomorphic to a product space, i.e. trivial.

As an example to a locally constant sheaf arising from a covering space which is not constant, consider the double cover of the circle by itself (draw the picture, see where it fails!). Locally, the sheaf obtained is the constant sheaf, but there are no global sections; Given a section $f : S^1 \to S^1$, note that $f(S^1)$ is not open in S^1, and thus f cannot be continuous. To make this precise, take $f^{-1}(U)$ where U is an open neighborhood of the boundary point of $f(S^1)$, and see it is not open. Thus it cannot be the constant sheaf. □

Remark 7.8. *Note that Exercise 7.7 shows that the picture for a σ-compact ℓ-space is very different from the case of a locally connected space. For the first* *every locally constant sheaf was constant, and for the latter every non-trivial covering space gives rise to a locally constant sheaf which is not constant.*

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Exercise 8.1. *Let X be a smooth or an F-analytic manifold. Show that $\overline{C_c^\infty(X)}^w = C^{-\infty}(X)$.*

Proof. Recall that $C^{-\infty}(X) = \mu_c^\infty(X)^*$. Given a topological vector space V, for $W \subseteq V^*$ the space W is dense w.r.t the weak topology if and only if $W^\perp = \{v \in V : \langle w, v\rangle = 0 \ \forall w \in W\} = \{0\}$. To see the relevant direction, if $W^\perp = \{0\}$, we will show that for every $\xi \in V^*$, finite set $S \subset V$ and $\epsilon < 0$ we can find $w \in W$ such that $\xi_{|S} = w_{|S}$. Given such $\xi \in V^*$, $S = \{v_1, \ldots, v_n\}$ and $\epsilon > 0$, assume S is a linearly independent set, and consider $\rho : V^* \to \mathbb{R}^n$ by $\rho(\eta) = (\langle \eta, v_1\rangle, \ldots, \langle \eta, v_n\rangle)$. The map $\rho_{|W}$ is onto, since otherwise there exists some $c_i \in \mathbb{R}$ such that $\sum_{i=1}^n c_i\langle w, v_i\rangle = 0$ for all $w \in W$ (it must lie in some hyperplane, and all hyperplanes are of this form), but this means that $\langle w, \sum_{i=1}^n c_i v_i\rangle = 0$ implying $\sum_{i=1}^n c_i v_i \in W^\perp = \{0\}$. The surjectivity of $\rho_{|W}$ allows us to find the desired $w \in W$. Thus it is enough to show that given $\eta \in \mu_c^\infty(X)$, if $\langle f, \eta\rangle = 0$ for all $f \in C_c^\infty(X)$ then $\eta = 0$.

Assume M is a smooth manifold. Given a non-zero measure η, there exists some $\mathbb{R}^n \simeq U \subset X$ such that $\eta_{|U} \neq 0$, to see this either use the fact that distributions form a sheaf, or view it as a positive function on Borel sets. Now, since $U \simeq \mathbb{R}^n$ we must have that $\eta_{|U} = g \cdot \mu_{Haar}$ where $g \in C^\infty(\mathbb{R}^n)$. Taking some cutoff function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_{|B_1(0)} \equiv 1$ and $\psi > 0$ implies the desired result as $\langle g\psi, \eta\rangle = \langle g\psi, g \cdot \mu_{Haar}\rangle = \langle g^2\psi, \mu_{Haar}\rangle > 0$ as this is an integral of a positive function.

For an F-analytic manifold we do the same procedure only this time ψ is the indicator function of the open unit ball in F^n. ☐

Definition 8.2. *Let M a smooth manifold and E a smooth real bundle over it. We define the topology on $C_c^\infty(M, E)$ by taking a cover $\{U_\alpha\}$ which locally trivializes both M and E, and announcing that a set is open if its preimage in $\bigoplus C_c^\infty(U_\alpha, E_{|U_\alpha})$ is open.*

Exercise 8.3. *Assuming the topology on $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$ was already determined, show that the topology on $C_c^\infty(M, E)$ is well defined.*

Proof. We need to show that given a different cover $\{V_\beta\}$ of M which locally trivializes M and E, we get the same topology.

Consider the cover $\{W_{\alpha,\beta}\}$ for $W_{\alpha,\beta} = U_\alpha \cap V_\beta$ which refines both covers. We need to show that for the addition map,

$$\bigoplus_{\alpha\in I}\bigoplus_{\beta\in J} C_c^\infty(W_{\alpha,\beta}, E_{|W_{\alpha,\beta}}) \xrightarrow{+} \bigoplus_{\alpha\in I} C_c^\infty(U_\alpha, E_{|U_\alpha})$$

a set in the range is open if and only if its preimage is open, where $W_{\alpha,\beta} \subseteq U_\alpha \simeq \mathbb{R}^n$ and $E_{|W_{\alpha,\beta}} \simeq E_{|U_\alpha} \simeq \mathbb{R}^k$. In order to show the above, it is enough to handle each case $\bigoplus_{\beta\in J} C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k) \xrightarrow{+} C_c^\infty(U_\alpha,\mathbb{R}^k) \simeq C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$ separately, since in the direct sum topology a set is open if all the injections $D_i \hookrightarrow \bigoplus D_i$ are continuous (for us - a convex set). Given a basic open set $U_{(L_m,\epsilon_m,B_m)} \subseteq C_c^\infty(U_\alpha,\mathbb{R}^k)$ where L_m are mixed differentiations, $\epsilon_m \in \mathbb{R}_{>0}$ and B_m are compact sets such that $\bigcup_{m=1}^{\infty} B_m = \mathbb{R}^n$, it is of the form $U_{(L_m,\epsilon_m,B_m)} = \sum_{m\in\mathbb{N}} V_{L_m,\epsilon_m,B_m}$, where

$$V_{L_m,\epsilon_m,B_m} = \left\{ f \in C^\infty(\mathbb{R}^n,\mathbb{R}^k) : \operatorname{supp}(f) \subseteq B_m,\ \sup_{x\in\mathbb{R}^n} ||L_m(f)|| < \epsilon_m \right\}.$$

Now, take a finite sum $\sum f_\beta \in +^{-1}(U_{\{L_m,\epsilon_m,B_m\}})$ for $\sum f_\beta = f = \sum_{i=0}^{l} f_{m_i}$ and $f_{m_i} \in V_{L_{m_i},\epsilon_{m_i},B_{m_i}}$. Set by $f_\beta = pr_\beta(f)$ the projection of f into $C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)$, and define $N = \#\{\beta : pr_\beta(f) \neq 0\}$ and $\epsilon'_{m_i} = \frac{\epsilon_{m_i} - \sup ||L_{m_i}(f_{m_i})||}{N}$ and set $\epsilon'_m = \frac{\epsilon_m}{N}$ if $m \neq m_i$ for all $0 \leq i \leq l$. For $B'_{m,\beta} \subseteq W_{\alpha,\beta}$, compact sets which exhaust $W_{\alpha,\beta}$ and such that $B'_{m,\beta} \subseteq B_m$, the sets $U_{(L_m,\epsilon'_m,B'_{m,\beta})}$ are basic open sets in each $C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k)$, and their direct sum is open in the direct sum. Now, we claim that,

$$f \in \bigoplus_{\beta: f_\beta\neq 0} f_\beta + U_{(L_m,\epsilon'_m,B'_{m,\beta})} \subseteq +^{-1}(U_{(L_m,\epsilon_m,B_m)}).$$

Given $g = \sum_{\beta: f_\beta \neq 0} g_\beta$ where $g_\beta \in U_{(L_m,\epsilon'_m,B'_{m,\beta})}$, then $g_\beta = \sum_{i_\beta=1}^{l_\beta} g_{\beta,i_\beta}$ where $g_{\beta,i_\beta} \in V_{L_{n_{i_\beta}},\epsilon'_{n_{i_\beta}},B'_{n_{i_\beta},\beta}}$.

Thus if $n_{i_\beta} = m_i$ for some i, we have $\sup_{x\in B_{m_i}} ||L_{m_i}(g_{\beta,m_i})|| < \epsilon'_{m_i} = \frac{\epsilon_{m_i} - \sup ||L_{m_i}(f)||}{N}$ implying that,

$$\begin{aligned}
\sup_{x\in B_{m_i}} \Big|\Big| \sum_{\beta: f_\beta\neq 0} L_{m_i}(f_{m_i,\beta} + g_{\beta,m_i}) \Big|\Big| &\leq \sup_{x\in B_{m_i}} \Big|\Big| \sum_{\beta: f_\beta\neq 0} L_{m_i}(f_{m_i,\beta}) \Big|\Big| + \sum_{\beta: f_\beta\neq 0} \sup_{x\in B_{m_i}} ||L_{m_i}(g_{\beta,m_i})|| \\
&< \sup_{x\in B_{m_i}} \Big|\Big| L_{m_i}(f_{m_i}) \Big|\Big| + \sum_{\beta: f_\beta\neq 0} \Big(\frac{\epsilon_{m_i} - \sup ||L_{m_i}(f_{m_i})||}{N} \Big) \\
&= \epsilon_{m_i}.
\end{aligned}$$

Otherwise, if $n_{i_\beta} \neq m_i$ for all i, set $n' = n_{i_\beta}$, and using the requirement $\sup_{x\in B_{n'}} ||L_{n'}(g_{\beta,n'})|| < \frac{\epsilon_{n'}}{N}$ we note that:

$$\sup_{x\in B_{n'}} \Big|\Big| \sum_{\beta: f_\beta\neq 0} L_{n'}(g_{\beta,n'}) \Big|\Big| \leq \sum_{\beta: f_\beta\neq 0} \sup_{x\in B_{n'}} ||L_{n'}(g_{\beta,n'})|| < N\frac{\epsilon_{n'}}{N} = \epsilon_{n'}.$$

This allows us to conclude that $f + g = \sum_{\beta: f_\beta\neq 0} \sum_{i=1}^{l} f_{m_i,\beta} + \sum_{\beta: f_\beta\neq 0} \sum_{i_\beta=1}^{l_\beta} g_{\beta,i_\beta}$ lie in $U_{(L_m,\epsilon_m,B_m)} = \sum_{m\in\mathbb{N}} V_{L_m,\epsilon_m,B_m}$ for all such functions g, implying that the addition is continuous. For a less cumbersome approach, note that the embeddings $\bigoplus_{\beta\in J} C_c^\infty(W_{\alpha,\beta},\mathbb{R}^k) \to \bigoplus_{\beta\in J} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$ are continuous (a cookie for the person who finds a quick proof for this), so it is enough to show that the addition map $\bigoplus_{\beta\in J} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k) \xrightarrow{+} C_c^\infty(\mathbb{R}^n,\mathbb{R}^k) \simeq \bigoplus_{i=k}^{k} C_c^\infty(\mathbb{R}^n)$ is continuous. Since the domain has the direct sum topology, it is enough to check this for a finite direct sum, which follows by the continuity of addition in a topological vector space.

To show the map is open, it is enough to consider $\bigoplus_{\beta\in J} C_{K,c}^\infty(W_{\alpha,\beta},\mathbb{R}^k) \xrightarrow{+} C_K^\infty(\mathbb{R}^n,\mathbb{R}^k) \simeq \bigoplus_{j=1}^{k} C_K^\infty(\mathbb{R}^n)$, for every compact K, and since the domain has the direct sum topology and the basic open sets are finite sums of open sets in each coordinate, it is enough to show it for a finite direct sum $\bigoplus_{i=1}^{m} C_{K,c}^\infty(W_i,\mathbb{R}^k) \xrightarrow{+} \bigoplus_{j=1}^{k} C_K^\infty(\mathbb{R}^n)$ where $K \subset \bigcup W_i$. Now, use partition of unity f_i, with $C_i = \operatorname{supp}(f_i) \subset W_i$ where $\sum_{i=1}^{m} f_{i|_K} \equiv 1$ to get an onto map via the composition,

$$\bigoplus_{i=1}^{m} C_{K\cap C_i}^\infty(W_i,\mathbb{R}^k) \hookrightarrow \bigoplus_{i=1}^{m} C_{K,c}^\infty(W_i,\mathbb{R}^k) \xrightarrow{+} C_K^\infty(\mathbb{R}^n).$$

Since this is a continuous surjective map of Fréchet spaces, it must be open, implying that the addition is open since the embedding is continuous. □

8.1 Operations on distributions

Definition 8.4. *Let $\varphi : X \to Y$ be map of manifolds (either smooth or F-analytic) and set,*

$$Dist(X)_{\mathrm{prop},\varphi} = \{\xi \in Dist(X) : \varphi_{|_{\operatorname{supp}(\xi)}} \text{ is proper}\}.$$

1. *If φ is proper, define the pushforward $\varphi_*(\xi) \in Dist(X)$ for $f \in C_c^\infty(Y)$ by*
$$\langle \varphi_*(\xi), f\rangle = \langle \xi, \varphi^*(f)\rangle.$$
2. *For $\xi \in Dist(X)_{\mathrm{prop},\varphi}$ or $\xi \in Dist_c(X)$ and $f \in C_c^\infty(Y)$ we define the pushforward by*
$$\langle \varphi_*(\xi), f\rangle = \langle \xi, \rho_f \cdot \varphi^*(f)\rangle$$
where $\rho_f \in C_c^\infty(X)$ is a cut off function such that $\rho_{f|_U} \equiv 1$ and $\varphi^{-1}(\operatorname{supp}(f)) \cap \operatorname{supp}(\xi) \subset U$ is an open set containing $\operatorname{supp}(f\circ\varphi)$.

Exercise 8.5. *Show that the definition above for pushing forward distributions in $Dist(X)_{\mathrm{prop},\varphi}$ is well defined.*

Proof. First, note that $\rho_f\varphi^*(f)$ is indeed compactly supported in X since

$$\operatorname{supp}(\rho_f\varphi^*(f)) = \operatorname{supp}(\rho_f) \cap \varphi^{-1}(\operatorname{supp}(f)) = \operatorname{supp}(\xi) \cap \varphi^{-1}(\operatorname{supp}(f))$$

is compact by the demand that $\varphi_{|_{\operatorname{supp}(\xi)}}$ is proper.

The definition is independent of ρ_f, since given a different cutoff function ψ_f, then $(\rho_f - \psi_f)_{|V} \equiv 0$, where $\operatorname{supp}(\xi\varphi^*(f)) \subset V$ is an open set, implying $(\rho_f - \psi_f) \in C_c^\infty(X\backslash \operatorname{supp}(\xi\varphi^*(f)))$ and thus $\langle \xi\varphi^*(f), \rho_f - \psi_f\rangle = 0 \Rightarrow$

$\langle \xi, (\rho_f)\varphi^*(f)\rangle = \langle \xi, (\psi_f)\varphi^*(f)\rangle$. Finally, note that we can indeed find such ρ_f by taking a compact, smooth partition of unity w.r.t a finite a cover U_i of K. The proof for the F-analytic case is analogous. $\square$

Exercise 8.6. *Let $\varphi : X \to Y$ be map of manifolds (either smooth or F-analytic), such that $\varphi_{|C} : C \to Y$ is proper for some $C \subset X$. Show that there exists an open $C \subset U$ and smooth $\rho : X \to \mathbb{R}$ such that $\rho_{|U} \equiv 1$*

Proof. ==Warning. Requires fixing, construction is probably true, proof is false.==
For every $y \in Y$ by local compactness we can find $y \in V_y \subset K_y \subset Y$ where V_y is open and K_y compact. Construct a non-negative cutoff function $\rho_y : X \to \mathbb{R}$ such that for the compact set $\varphi^{-1}(K_y) \cap C$ we have that ${\rho_y}_{|_{\varphi^{-1}(K_y)\cap C}} \equiv 1$ and $\rho_y \in C_c^\infty(X)$. Now, $\bigcup_{y\in Y} V_y = Y$, and take a partition of unity $\text{supp} f_y \subseteq V_y$ with respect to the cover V_y (we can take both covers to be of countable size and thus obtain countably many functions in the partition of unity). Also, note that $f_y \in C_c^\infty(Y)$ since $\text{supp}(f_y) \subset K_y$ is closed and hence compact. Finally, set $\rho = \sum_{i\in\mathbb{N}} \rho_{y_i} f_i \circ \varphi$ and note that indeed $\rho_{|C} \equiv 1$. To finish, we have that,

$$\varphi^{-1}(K) \cap C \subseteq \varphi^{-1}(K) \cap \text{supp}(\rho) = \bigcup_{i\in\mathbb{N}} \text{supp}(\rho_{y_i}) \cap \varphi^{-1}(K \cap \text{supp}(f_i))$$
$$\subseteq \bigcup_{i\in\mathbb{N}} \text{supp}(\rho_{y_i}) \cap \varphi^{-1}(K \cap V_{y_i}).$$

But since we also have that, $\varphi^{-1}(V_y) \cap C \subseteq \text{supp}(\rho_y)$:

$$\bigcup_{i\in\mathbb{N}} \text{supp}(\rho_{y_i}) \cap \varphi^{-1}(K \cap V_{y_i}) = \bigcup_{i\in\mathbb{N}} C \cap \varphi^{-1}(V_{y_i} \cap K) = C \cap \bigcup_{i\in\mathbb{N}} \varphi^{-1}(V_{y_i} \cap K) = \varphi^{-1}(K) \cap C,$$

and we are done. $\square$

9 Targil 10

9.1 Pontryagin duality and Fourier transform

Let G be an abelian locally compact group. Define its Pontryagin dual by,

$$G^\vee = \{\chi : G \to U_1(\mathbb{C}) = S^1 \subset \mathbb{C} : \chi(g_1 g_2) = \chi(g_1)\chi(g_2),\ \chi \text{ is cts}\}.$$

The topology on $G^\vee$ is the compact open topology, i.e. a sub-basis for the topology is comprised of sets $M(K, V) = \{\chi \in G^\vee : \chi(K) \subseteq V\}$ where $K \subseteq G$ is compact and $V \subseteq S^1$ is open.

Theorem 9.1. *For a locally compact abelian group G, we have that $G^{\vee\vee} \simeq G$.*

Exercise 9.2. *Let G be a locally compact, Hausdorff abelian group, then $G^\vee$ is a locally compact Hausdorff abelian group.*

Proof. (See Mathstackexchange Q. 1502405) We see that characters form an abelian group. Since S^1 is a topological group, the compact open topology on $G^\vee$ is equivalent to the topology of uniform convergence on compact sets.

Thus, in order to show that the multiplication and inverse are continuous it is enough to show that if $f_n \to f$ and $g_n \to g$ uniformly on compact sets then $f_n \cdot g_n^{-1} \to f \cdot g^{-1}$ uniformly on compact sets. Now, if $K \subset G$ is compact, note that this follows from the following bound ($\forall x \in K$):

$$|f_n g_n^{-1} - f g^{-1}| \leq |f_n(g_n^{-1} - g^{-1})| + |(f_n - f)g^{-1}| = |g_n - g| + |f_n - f|.$$

Now to show it is locally compact, consider the space $(S^1)^G$ of all functions $f : G \to S^1 \simeq \mathbb{R}/\mathbb{Z}$ with the product topology (i.e. a basis is given by open sets in only finitely many components). It is a compact space by Tychonoff's theorem, and it has the space

$$\tilde{G} = \bigcap_{g_1, g_2 \in G} \{\chi : G \to S^1 : \chi(g_1 g_2) = \chi(g_1)(g_2)\},$$

as a closed subspace, implying that $\tilde{G}$ is compact. Furthermore, for every $S \subseteq G$ and $\epsilon > 0$ the set $A(S, \epsilon) = \{\chi \in (S^1)^G : \chi(S) \subseteq [-\epsilon, \epsilon]\}$ is also closed and compact in $(S^1)^G$ as the complement is a union of sets of the form $\{\chi : G \to S^1 : \chi(s) \in [-\epsilon, \epsilon]^c\}$ for some $s \in S$, which are open.

In particular, taking an open neighborhood $e \in U \subset G$ the sets $V(U, \epsilon) = A(U, \epsilon) \cap \tilde{G}$ are closed and compact in $(S^1)^G$. Take $0 < \epsilon < \frac{1}{2}$, we show that we have that $V(U, \epsilon) \subseteq G^\vee$. Start with an open $e \in U_0 = U \subset G$, and choose a sequence of neighborhoods (U_n) such that $U_{n+1} \cdot U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$ and set $\epsilon_n = \frac{\epsilon}{2^n}$. Taking $\chi \in V(U_n, \epsilon_n)$, we see that since for $x \in U_{n+1}$ we have that $\chi(x) \in [-\epsilon_n, \epsilon_n]$ and $x^2 \in U_n$ we get $\chi(x^2) = \chi(x)^2 \in [-\epsilon_n^2, \epsilon_n^2] \subseteq [-\frac{\epsilon_n}{2}, \frac{\epsilon_n}{2}]$, implying that $V(U_n, \epsilon_n) \subseteq V(U_{n+1}, \epsilon_{n+1})$.

Now, take $\chi \in V(U, \epsilon)$ and a basic open set $(-\delta, \delta) \subset S^1$ for $\delta > 0$. We have that $[-\epsilon_n, \epsilon_n] \subseteq (-\delta, \delta)$ for n big enough, implying that $e \in U_n \subseteq \chi^{-1}((-\delta, \delta))$ which means that χ is continuous at e. Since χ is a homomorphism, we can show it is continuous everywhere; if $\chi(g) \in W \subset S^1$ and W is open, we have that $(-\delta, \delta) \subseteq \chi(g^{-1})W$ for some $\delta > 0$ and that,

$$\begin{aligned}\chi^{-1}(\chi(g^{-1})W) &= \{y \in G : \chi(y) \in \chi^{-1}(g)W\} = \{y \in G : \chi(gy) \in W\} \\ &= g^{-1}\{gy \in G : \chi(gy) \in W\} = g^{-1}\chi^{-1}(W).\end{aligned}$$

Now, for some $m \in \mathbb{N}_0$ big enough, the following implies that $g \in gU_m \subseteq \chi^{-1}(W)$:

$$U_m \subset \chi^{-1}(\chi(g^{-1})W) = g^{-1}\chi^{-1}(W).$$

We know that $V(U, \epsilon)$ is compact in the product topology, and want to show it is compact with respect to the compact open topology. For this, it is enough to show that any net in $V(U, \epsilon)$ has a converging subnet in the compact open topology. Assume we are given some net $(x_\alpha) \in V(U, \epsilon)$, then it has a subnet $(f_\beta) \to f$ converging in the product topology with $f_\beta, f \in V(U, \epsilon)$. Now, note that $V(U, \epsilon)$ is uniformly equicontinuous, that is if $g_1, g_2 \in G$ and $g_1 g_2^{-1} \in U_n$ then for any $\chi \in V(U, \epsilon)$,

$$|\chi(g_1) - \chi(g_2)| = |\chi(g_1)\chi^{-1}(g_2) - 1| = |\chi(g_1 g_2^{-1}) - 1| \leq \epsilon_n.$$

Given a basic open neighborhood of the identity character $1_G \in M(K, B_{\epsilon'}(0))$, where K is compact, for every $g \in K$ we have that $g \in U_n g$ (for n big enough).

Now, taking any $g' \in U_n g$, we get that $gg'^{-1} \in U_n$ implying that for some big enough β we have that $|f(g) - f_\beta(g)| < \epsilon_n$ and that,

$$|f(g') - f_\beta(g')| \leq |f(g') - f(g)| + |f_\beta(g') - f_\beta(g)| + |f(g) - f_\beta(g)| < 3\epsilon_n.$$

Taking $n > n_g$ such that $\epsilon_n < \frac{\epsilon'}{3}$, we see that $f_\beta \to f$ uniformly on $U_n g$, but since K is compact we can cover it with finitely many sets of the form $U_{n_g} g$, and take $n = \max_{0 \leq i \leq k} \{n_{g_i}\}$ and appropriate β.

To finish off the argument, note that by locally compactness every $g \in G$ has a neighborhood $g \in U$ with compact closure $U \subset K$, and we have that $M(K, B_0(\epsilon)) \subseteq V(U, \epsilon)$ for an appropriate $\frac{1}{2} > \epsilon > 0$. □

Exercise 9.3. *Let G be a locally compact, Hausdorff abelian group. Show that if G is compact then $G^\vee$ is discrete, and that if G is discrete then $G^\vee$ is compact.*

Proof. If G is compact, take $P = \{\theta : -\frac{1}{4} < \theta < \frac{1}{4}\} \subset \mathbb{R}/\mathbb{Z}$, and consider the open set $1_G \in M(G, P)$. Since the only subgroup in $P \subset S^1$ is $\{1\}$, we have that $M(G, P) = \{1_G\}$ is open, implying that every $\chi \in G^\vee$ is open and hence $G^\vee$ discrete.

If G is discrete, every character is continuous and since $G^\vee \subseteq \tilde{G}$, where $\tilde{G}$ is the space of all homomorphisms we have that $G^\vee = \tilde{G}$. Since $G^\vee = \tilde{G}$ is a closed space of the compact space $(S^1)^G$ it is compact in the product topology. Finally, the map $\mathrm{Id} : G^\vee_{prod} \to G^\vee_{c.o}$ is continuous since given a compact $K \subset G$, it must be a finite number of points implying that $M(K, V) = \bigcap_{i=0}^n M(\{s_i\}, V)$, which is open in the product topology, implying that its image is a compact set. □

Exercise 9.4. *Let G be a locally compact, Hausdorff abelian group, and $H \leq G$ a closed subgroup.*

1. *Show that Pontryagin duality is a contravariant endofunctor in the category of locally compact abelian groups.*
2. *Show that $H^\vee \simeq G^\vee / H^\perp$ where $H^\perp = \{\chi \in G^\vee : \chi(h) = 0 \ \forall h \in H\}$, and that if H and H are vector spaces then this is a homeomorphism.*

Proof. For the first item, we know by Exercise 9.2 that $G^\vee$ is a locally compact abelian group. Given a continuous homomorphism $\varphi : G \to G'$, it gives rise to a homomorphism $\varphi^\vee : G'^\vee \to G^\vee$ via precomposition, i.e. $\varphi^\vee(\chi)(g) = \chi \circ \varphi(g)$. Since $\mathrm{id_G}^\vee = \mathrm{id}_{\mathrm{G}^\vee}$, and $(\varphi_1 \circ \varphi_2)^\vee = \varphi_2^\vee \circ \varphi_1^\vee$ for appropriate homomorphisms φ_1 and φ_2, it is left to show that it is continuous. This follows since for compact $K \subset G$ and open $U \subset S^1$ we have that,

$$\varphi^{\vee -1}(M_G(K, U)) = \{\chi \in G'^\vee : \chi \circ \varphi(K) \subset U\} = M_{G'}(\varphi(K), U),$$

and since φ is continuous $\varphi(K)$ is compact and $M_{G'}(\varphi(K), U)$ is a basic open set in $G'^\vee$.

For the second item note that we have the inclusion $i : H \hookrightarrow G$ which gives rise to the surjective projection $p : G^\vee \to H^\vee$ (Pontryagin duality is an equivalence of categories). Since $\ker(p) = \{\chi \in G^\vee : \chi(h) = 0 \ \forall h \in H\} = H^\perp$, by Noether's isomorphism theorem we have that $H^\vee \simeq G^\vee / H^\perp$ as groups. To see this is indeed a homeomorphism, it is enough to show that p is an open map (the quotient map $q : G^\vee \to G^\vee / H^\perp$ is continuous and open as

$q^{-1}(q(M_G(K,U))) = M_G(K,U)H^{\perp})$, but this holds since for a basic open set $M_G(K,U)$ we have that $p(M_G(K,U)) = (M_H(K\cap H,U))$, and $H\cap K$ is compact since H is closed. Note that for the last equality we use a version of the Hahn-Banach theorem. □

Remark 9.5. *Note that we can identify $\mathbb{R}^n$ with $(\mathbb{R}^\vee)^n$ by $x(\xi) = e^{-i\xi\cdot x}$ for $\xi \in \mathbb{R}^n$ and $\xi \cdot x$ is the dot product.*

Definition 9.6. *Let V be a topological vector space over a local field. We define $\mathcal{G}(V) := S^*(V, Haar(V))$.*

Definition 9.7. *We define the Fourier transform in several steps.*

1. *Firstly, for a vector space V (either Archimedean or non-Archimedean) define $\mathcal{F} : \mu_c(V) \to C(V^\vee)$ by $\mathcal{F}(\mu)(\chi) = \int \chi d\mu$ (Exercise - $\mathcal{F}(\mu)$ is continuous).*
2. *We note that for the subspace $\mu_c^\infty(V) \subset \mu_c(V)$ we have that $\mathcal{F}(\mu_c^\infty(V)) \subset S(V^\vee)$.*
3. *Since we also have that $\mu_c^\infty(V) \subset S(V, Haar(V))$, and it is dense in $S(V, Haar(V))$, we would like to define the Fourier transform on $S(V, Haar(V))$ via continuity.*
4. *Finally, we define the Fourier transform $\mathcal{F} : S^*(V^\vee) \to \mathcal{G}(V) = S^*(V, Haar(V))$ via duality.*

For the second and third steps we solve the following exercise.

Exercise 9.8. *Show that the Fourier transform $\mathcal{F} : S(V, Haar(V)) \to S(V^\vee)$ is continuous for an Archimedean V and is indeed contained in $S(V^\vee)$.*

Proof. Assume V is a real vector space of dimension n, and recall that the topology on $S(V)$ is determined by the semi-norms $\|f\|_{\alpha,\beta} = \sup_{x\in V} |\Phi_\alpha(x)\frac{\partial^\beta f(x)}{\partial x^\beta}|$ where $\alpha, \beta \in \mathbb{N}_0^n$ and $\Phi_\alpha(x) = \prod_{j=1}^{n} x_j^{\alpha_j}$. It is enough to show that for every $f \in C_c^\infty(V, Haar(V))$ and semi-norm $\|\cdot\|_{\alpha,\beta}$ there exists a semi-norm $\|\cdot\|'$ such that $\|\mathcal{F}(f)\|_{\alpha,\beta} \le C\|f\|'$. Now, recall that,

$$\frac{i\partial\mathcal{F}(f)}{\partial\xi_j} = \int_{\mathbb{R}^n} \frac{i\partial}{\partial\xi_j}(e^{-i\xi\cdot x}f(x))dx = \mathcal{F}(x_j f),$$

where one can differentiate directly using the definition to verify the above procedure. The other side of the coin is given by integration by parts,

$$\xi_j\mathcal{F}(f) = \int_{\mathbb{R}^n} \xi_j e^{-i\xi\cdot x}f(x)dx = \left[-e^{-i\xi\cdot x}f(x)\right]_{-\infty}^{\infty} - \int_{\mathbb{R}^n} \frac{\xi_j}{-i\xi_j}e^{-i\xi\cdot x}\frac{\partial f(x)}{\partial x_j}dx = \mathcal{F}(\frac{-i\partial(f)}{\partial x_j}).$$

Note that since the functions $e^{-i\xi\cdot x}$ converge weakly to zero as distributions as $|\xi| \to \infty$ this shows that smooth, compactly supported measures are mapped into $S(V^\vee)$. We can now bound $\mathcal{F}(f)$ properly using the above relations:

$$\|\mathcal{F}(f)\|_{\alpha,\beta} = \sup_{x^\vee \in V^\vee} \left| \Phi_\alpha(x^\vee) \frac{(-i\partial)^\beta \mathcal{F}(f)(x^\vee)}{\partial (x^\vee)^\beta} \right| = \sup_{x^\vee \in V^\vee} \left| \int_V x^\vee \frac{(-i\partial)^\alpha (\Phi_\beta(-x) f)}{\partial x^\alpha} d\mu(x) \right|$$
$$\leq \sup_{x^\vee \in V^\vee} \int_V \left| x^\vee \frac{\partial^\alpha (\Phi_\beta(-x) f)}{\partial x^\alpha} \right| d\mu(x) \leq C \sup_{x \in V} \left((1+|x|)^{n+1} \left| \frac{\partial^\alpha (\Phi_\beta(|x|) f)}{\partial x^\alpha} \right| \right).$$

where $C = \int_V \frac{1}{(1+|x|)^{n+1}} d\mu(x)$. Since the last expression is a linear combination of norms of the form $\|f\|_{\alpha',\beta'}$ for $|\alpha'| \leq |\alpha|+n+1$ and $|\beta'| \leq |\beta|$, this implies that $\mathcal{F}$ is continuous. Note that we can also use this to show that $\mathcal{F}(f)$ is Schwartz, since if all the norms $\|\cdot\|_{\alpha,\beta}$ are bounded then the value of $|\Phi_\alpha(x) \frac{\partial^\beta f(x)}{\partial x^\beta}|$ decays to 0 as $|x| \to \infty$ for every α and β. ☐

Definition 9.9. *Let F be a local field and $\chi : F^\times \to \mathbb{C}^\times$ a character. We define a functor for a 1-dimensional space V over F by*

$$\chi(V) := \{\varphi : V^* \to \mathbb{C} : \varphi(\alpha v) = \chi(\alpha)\varphi(v) \forall \alpha \in F^\times, v \in V^*\}.$$

For the character $x \mapsto x^\alpha$ where $\alpha \in \mathbb{Q}^\times$ we get for $V/\mathbb{R}$ the space V^α.

Remark 9.10. *Pushing forward a Schwartz measure along a submersion φ yields a Schwartz measure. If φ is a linear projection, this follows from Fubini's theorem.*

Exercise 9.11. *(Functoriality of Fourier transform) Let $W \subset V$ be vector spaces over a local field, and denote the inclusion of W in V by i, and set $p : V^\vee \to W^\vee$ for the induced linear map on the duals, then the following diagrams commute:*

$$\begin{array}{ccc} S(V) & \xrightarrow{\quad i^* \quad} & S(W) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ S(V^\vee, Haar(V^\vee)) & \xrightarrow{p_*} & S(W^\vee, Haar(W^\vee)) \end{array} \qquad \begin{array}{ccc} S^*(V) & \xleftarrow{i_*} & S^*(W) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{G}(V^\vee) & \xleftarrow{p^*} & \mathcal{G}(W^\vee), \end{array}$$

note that this is possible since p is a submersion (linear and surjective) so pushing Schwartz measures along it yields Schwartz measures.

Proof. We start by showing the right hand side diagram commutes. Since i_*, the Fourier transform and p^* are continuous with respect to the weak topology, it is enough to prove commutativity for a dense set in $S^*(W)$.

First take the delta function $\delta_0 \in S^*(W)$, it is a compactly supported measure, and it holds that $i_*(\delta_0) = \delta_0$. Furthermore, since $\mathcal{F} : S^*(V) \to \mathcal{G}(V^\vee)$ is defined via duality we have that $\mathcal{F}(\delta_0) = 1$:

$$\langle \mathcal{F}(\delta_0), f\mu \rangle = \langle \delta_0, \mathcal{F}(f\mu) \rangle = F(f\mu)(0_{V^{\vee\vee}}) = \int_{V^\vee} f d\mu = \langle 1, f\mu \rangle,$$

where the third equality is sensible since $\mathcal{F}(f\mu) \in S(V^{\vee\vee})$ and $0_{V^{\vee\vee}}(\chi) = 1$ for all $\chi \in V^\vee$. We can also show that $p^*(1) = 1$. Consider $\mathcal{G}(W^\vee)$ as a subspace of $C^{-\infty}(W^\vee)$, there the generalized Schwartz function 1 is a smooth function, and note that the following diagram, where the horizontal arrows are the inclusions is commutative:

$$\begin{array}{ccccc}
\mathcal{G}(V^\vee) & \hookrightarrow & C^{-\infty}(V^\vee) & \hookleftarrow & C^\infty(V^\vee) \\
\downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\
\mathcal{G}(W^\vee) & \hookrightarrow & C^{-\infty}(W^\vee) & \hookleftarrow & C^\infty(W^\vee).
\end{array}$$

Now, note that every measure $f\mu \in \mu_c^\infty(V^\vee)$ can be treated either as a functional on smooth functions (since it has compact support as a distribution), or as the parameter a generalized function takes values on. This is utilized in the third equality bellow to yield the required result:

$$\langle p^*(1), f\mu\rangle = \langle p^*_{C^{-\infty}}(1), f\mu\rangle = \langle 1, p_*(f\mu)\rangle = \langle p_*(f\mu), 1\rangle = \langle f\mu, p^*_{C^\infty}(1)\rangle = \langle 1, f\mu\rangle.$$

Note that since p_* is a submersion pushing forward a supported smooth measure along it yields a smooth measure.

Since δ_w for any other $w \in W$ is just a translation of δ_0 by w, its Fourier transform is $\mathcal{F}(\delta_w)(\chi) = \chi(w)$, and i_* and p^* are invariant to translations, the diagram is commutative for delta distributions. The space of Delta distributions $\mathrm{span}_{\mathbb{R}}\{\delta_w\}_{w\in W}$ is dense w.r.t the weak topology since for every function f with $f(x_0) \neq 0$ we can take suitable $c \in \mathbb{R}$ such that $|\langle \xi - c\delta_x, f\rangle|$ is small as desired.

To see this implies the commutativity of the left diagram, it is enough to show that if $A^* = 0$ for $A^* : V_2^* \to V_1^*$ where A^* is the dual map to the linear map $A : V_1 \to V_2$, then $A = 0$, and use this for $\mathcal{F}i_* - p^*\mathcal{F}$. If $A^* = 0$, we have for every $\xi_2 \in V_2^*$ and $v_1 \in V_1$ that $0 = \langle A^*\xi_2, v_1\rangle = \langle \xi_2, Av_1\rangle$. If there exists $v_1 \in V_1$ such that $Av_1 \neq 0$, then we can define a non-zero linear functional $\xi : \mathrm{span}_{\mathbb{R}}\{Av_1\} \to \mathbb{R}$ via $\langle \xi, Av_1\rangle = 1$, and extend it to a non-zero continuous functional $\xi_2 \in V_2^*$ by the Hahn-Banach theorem. This yields a contradiction as

$$1 = \langle \xi_2, Av_1\rangle = \langle A^*\xi_2, v_1\rangle = \langle 0, v_1\rangle = 0.$$

□

10 Targil 11-12

10.1 Wave front set

Motivated by the definition of Fourier transform for Schwartz distributions and hence of compactly supported distributions, we move to study smoothness of distributions. Roughly speaking, our intuition will be the following (following Hormander's philosophy in [2, Chapter VIII, Page 251]). A compactly supported distribution ξ is said to be smooth if it is a smooth function. Morally speaking the Fourier transform interchanges between smoothness and rapid decay, and we can show that a compactly supported distribution is a smooth function if its Fourier transform is a rapidly decaying function. If the Fourier transform of ξ is not a rapidly decreasing function, we can use the directions in $V^\vee \simeq V^* \otimes F^\vee \simeq V^*$ (choose a character $\varphi : F \to S^1$) in which it does not rapidly decay to get information on the lack of smoothness of ξ. The data about the lack of smoothness of ξ will be encoded in the wave front set of ξ.

Definition 10.1. *Let X be an F-analytic manifold, W an F-vector space, F a non-Archimedean local field and $pr_X : X \times W \to X$ the standard projection. We set,*

$$S^W(X \times W) = \{f \in C^\infty(X \times W) : pr_{X|_{\operatorname{supp} f}} : \operatorname{supp}(f) \to X \text{ is proper}\}.$$

Definition 10.2. *Let X be a smooth manifold, W an F-topological vector space and F an Archimedean local field. We set $w^m = \prod_{i=1}^{\dim W} w_i^{m_i}$ and define $S^W(X \times W)$ to be:*

$$\{f \in C^\infty(X \times W) : \forall K \subset X, \forall m, n \in \mathbb{N}^{\dim W}, D \in \operatorname{Diff}(X), \|Df\|_{m,n,K} < \infty\},$$

where $\|Df(x,w)\|_{m,n,K} = \sup_{(x,w)\in K\times W} \left| D\frac{\partial^n f(x,w)}{\partial w_n} w^m \right|.$

Definition 10.3. *Let $f \in C^\infty(V)$ and $v \in V$ for a topological vector space V over a local field F. We say that f vanishes asymptotically along v if $\exists U \ni v$ open neighborhood such that $a^*f \in S^F(U \times F)$ where $a : U \times F \to V$ via $(u, \lambda) \mapsto \lambda u$. One can interpret this as f being Schwartz in (a conical neighborhood of a) direction v.*

Remark 10.4. *Definition 10.3 is equivalent to the following: for $x \in V$ there exists $\rho \in C_c^\infty(V)$ such that $\rho(x) \neq 0$ and $\rho a^*(f) \in S(V \times F)$.*

Example 10.5. *The function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = e^{-x^2}$ vanishes asymptotically for every $v \in \mathbb{R}^2$ not on the line $\{x = 0\}$.*

Proof. Take any $v = (x, y) \in \mathbb{R}^2$ such that $x \neq 0$. We can find an open ball denoted by B around v, small enough such that it doesn't intersect $\{x = 0\}$, we show that $a^*f \in S^{\mathbb{R}}(B \times \mathbb{R})$.

Given $K \subset B$, for every $m, n \in \mathbb{N}$ and $D \in \operatorname{Diff}(B)$ we have that $\|Da^*f\|_{m,n,K} = \sup_{(x,y,w)\in K\times\mathbb{R}} \left| D\frac{\partial^n a^*f}{\partial w_n} w^m \right| < \infty$ since $D(a^*f(x, y, w)) = D(f(xw, yw)) = D(e^{-w^2x^2})$ is a Schwartz function in the parameter w for every $x, y \in K$.

If $x = 0$, then for any neighborhood $v \in B$ and $K \subset B$ we can take $D = 1$ with $n = 0$ and $m = 1$ and get that $\lim_{w\to\infty} |wa^*f(0, y, w)| = |we^0| = \infty$, implying that f doesn't vanish asymptotically along $(0, y)$. ∎

Exercise 10.6. *Show that if $f \in C^\infty(V)$ vanishes asymptotically along 0 then $f \equiv 0$.*

Proof. If f vanishes asymptotically along 0 then there exists an open $0 \in U$ such that $a^*f \in S^F(U \times F)$. Choosing any neighborhood $0 \in K \subset U$, for every $x \in V$ we have that $\frac{x}{\alpha} \in K$ for all $\alpha > \alpha_0$ where we take α_0 big enough. Now, a^*f is constant on curves of the form $(\frac{x}{\alpha}, \alpha)$, if V is non-Archimedean then $\operatorname{supp}(a^*f) \cap pr^{-1}(K)$ is compact but $(\frac{x}{\alpha}, \alpha) \in \operatorname{supp}(a^*f) \cap pr_V^{-1}(K)$ for big enough α which is not compact, implying a contradiction, and so we have that $f(x) = 0$.

If V is Archimedean, then since $|a^* f w|$ must be bounded on $K \times F$, we get that $0 = \lim_{\alpha \to \infty} a^* f(\frac{x}{\alpha}, \alpha) = f(x)$, finishing the proof. □

Definition 10.7. *Let V be a vector space over a local field and $\xi \in C^{-\infty}(V)$, we say that ξ is smooth at $(x, w) \in V \oplus V^*$ if $\exists \rho \in C_c^\infty(V)$ with $\rho(x) \neq 0$ such that $\mathcal{F}(\rho\xi) \in C^\infty(V^\vee) \simeq C^\infty(V^*)$ vanishes asymptotically along w.*

Definition 10.8. *Let V be a vector space over a local field and $\xi \in C^{-\infty}(V)$, we define the wave front set of ξ by:*

$$WF(\xi) = V \oplus V^* \backslash \{(x, w) : \xi \text{ is smooth at } (x, w)\}.$$

Remark 10.9. *For a manifold M one defines $WF(\xi) \subset T^*M$ analogously, where now a distribution is smooth at (x, w) if it is smooth there with respect to some chart $x \in U \subset M$.*

Example 10.10. *Compute the wave front set of the Dirac delta function $\delta \in Dist(\mathbb{R}^n)$.*

Proof. Since $\text{supp}(\delta) = \{0\}$, we have that $WF(\delta) \subseteq \{0\} \times V^*$. This holds since for every $x \notin \text{supp}(\delta)$, we can take a bump function $\rho \in C_c^\infty(\mathbb{R}^n)$ which is non-zero at x and zero at 0, and since $\rho\delta = 0$, its Fourier transform is smooth in every direction $w \in (\mathbb{R}^n)^\vee \simeq \mathbb{R}^n$. For every $w \in V^*$ we get that $(0, w) \in WF(\delta)$ since $\mathcal{F}(\delta) = 1$ which doesn't vanish asymptotically in any direction. □

Definition 10.11. *Pushforward and pullback of sets. TBA (use diagram).*

Corollary 10.12. *Let $\varphi : M \to N$ be a map between manifolds, $\xi_M \in C^{-\infty}(M)$ and $\xi_N \in C^{-\infty}(N)$, then:*

1. *If φ is a submersion, then $WF(\varphi^*(\xi_N)) = \varphi^*(WF(\xi_N))$.*
2. *If $\xi_M \in C^{-\infty}_{prop,\varphi}(M)$, then $WF(\varphi_*(\xi_M)) \subseteq \varphi_*(WF(\xi_M))$.*

Exercise 10.13. *Show that if $\xi \in C_c^{-\infty}(V)$ is smooth at (v, l) for a given $l \in V^*$ and all $v \in \text{supp}(\xi)$ then $l_*(\xi) \in C^{-\infty}(\mathbb{R})$ is a smooth function with compact support.*

Proof. Note that since ξ is compactly supported, it is l-proper, $l_*(\xi)$ is compactly supported and we can use (2) of the previous corollary. Explicitly, we have that $WF(l_*(\xi)) \subseteq l_*(WF(\xi))$, we show that $l_*(WF(\xi)) = l(\text{supp}(\xi)) \times \{0\}$.

$l_*(WF(\xi))$ consists of all elements $(x, y) \in \mathbb{R} \oplus \mathbb{R}^*$ such that there exists $(v, w) \in WF(\xi)$ such that $l(v) = x$ and $(dl)_v^*(y) = w$ (draw the picture). Since l is a linear functional, $(dl)_v^*(y) = y \circ l \in V^*$, but since $y \in \mathbb{R}^*$, the functional $y \circ l$ is just given by λl for some $\lambda \in \mathbb{R}$. But as $WF(\xi)$ is conical, and ξ is smooth at (v, l), then $(v, l) \notin WF(\xi)$ and hence $(v, \lambda l) \notin WF(\xi)$ for all $\lambda \in \mathbb{R}$.

This implies that there does not exist $(v, w) \in WF(\xi)$ such that $(dl)_v^*(w) = y$ for all $y \in \mathbb{R}^*$ and in particular $l_*(WF(\xi)) = l(\text{supp}(\xi)) \times \{0\}$. Since smoothness is a local property, we can show an analogous property for manifolds (think about the generalization). □

Exercise 10.14. *Let $L \subseteq V$ be vector spaces over a local field F and $N \subset M$ F-manifolds (either smooth if F is Archimedean or F-analytic if F is non-Archimedean).*

1. *Compute $WF(i_*(\mu))$ where $\mu \in Haar(L)$ is a Haar measure on L.*
2. *Compute $WF(i_*(\eta))$ where η is a smooth measure on N and $i : N \hookrightarrow M$ is the embedding of N into M.*

Proof. 1. Note that the Haar measure μ is smooth when restricted to L, and denote by $i : L \hookrightarrow V$ the standard embedding. Since i is linear, we can use Corollary 10.12(2), we thus know that $WF(i_*(\mu)) \subseteq i_*(WF(\mu))$. Since $WF(\mu)$ is smooth on L, we have that $i_*(WF(\mu)) = i_*(L \times \{0\})$, which is exactly all those (v, w) such that $v \in \mathrm{Im}(i) = L$ and $(di)^*_v(w) = w \circ i = 0$, meaning that

$$i_*(WF(\mu)) = \{(v, w) \in V \oplus V^* : v \in L, \langle w, x\rangle = 0 \ \forall x \in L\} = CN_L^V.$$

We claim that $WF(i_*(\mu)) = CN_N^M$, this amounts to showing that $i_*(\mu)$ is not smooth at (x, w) for all $(x, w) \in L \times L^\perp$. Take $(x, w) \in L \times L^\perp$ and $\rho \in C_c^\infty(V)$ such that $\rho(x) \neq 0$, using Exercise 9.11 (p is a submersion) we see that

$$\mathcal{F}(\rho i_*(\mu)) = \mathcal{F}(i_*(\rho_{|_L}\mu)) = p^*\mathcal{F}(\rho_{|_L}\mu) = p^*(\mathcal{F}(\rho_{|_L}) * \mathcal{F}(\mu)) = p^*(\mathcal{F}(\rho_{|_L}) * \delta_0) = p^*(\mathcal{F}(\rho_{|_L})).$$

Since $\rho_{|_L}$ is smooth and compactly supported, its Fourier transform is a Schwartz function, and $p^*(\mathcal{F}(\rho_{|_L})) = \mathcal{F}(\rho_{|_L}) \circ p$, which is constant on the $L^\perp$ axis in $V \simeq L \times L^\perp$. In particular, $a^*\mathcal{F}(\rho_{|_L}) \circ p \notin S^F(U \times F)$ for all neighborhoods U of w.

2. Assume M and N are smooth manifolds. Let $x \in N$ and take a neighborhood $x \in U_x \simeq \mathbb{R}^m$ and a diffeomorphism $\varphi : U_x \to \mathbb{R}^m$ such that $\varphi(U_x \cap N) \simeq \mathbb{R}^n \subset \mathbb{R}^m$. Here, we arrive at the same situation as in 1. as a smooth measure is locally just a smooth function multiplied by a Haar measure, and we know that $WF(\varphi_* i_*(\eta)) = U_x \cap \mathrm{supp}(i_*(\eta)) \times (\mathbb{R}^n)^\perp \subseteq CN_{\mathbb{R}^n}^{\mathbb{R}^m}$. Now, by Hormander's theorem the wave front set is invariant to diffeomorphisms, i.e $WF(i_*(\eta)) = WF(\varphi_* i_*(\eta))$, and since smoothness is a local property, we get that $WF(i_*(\eta)) = \mathrm{supp}(i_*(\eta)) \times N^\perp \subset T^*M$. □

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