ULPANA PROJECT IN REP. THEORY

Exercise 1. *Any function can be written as a sum of even function and an odd function.*

Definition 2.

- *group*
- *group representation* (π, V)
  - $\pi : G \to GL(V)$
- *Morphism of representations*
- *sub representation*
- *irreducible representation*
  - $irr(G)$
- *Direct sum*

Exercise 3.

- *Schur's lemmas*
  - $Hom_G(\pi, \tau) = 0$ *if* $\pi \not\cong \tau$
  - $Hom_G(\pi, \pi) = \mathbb{C}$
- *complete reducibility*
  - $\pi \simeq \bigoplus_{\rho \in irr(G)} m_\rho \rho$
  - $\dim\mathrm{Hom}(\bigoplus_{\rho \in irr(G)} m_\rho \rho, \bigoplus_{\rho \in irr(G)} m'_\rho \rho) = \sum m_\rho m'_\rho$
  - *uniqueness of the decomposition.*
  - $\pi \cong \bigoplus_{\rho \in irr(G)} \mathrm{Hom}(\rho, \pi) \otimes \rho$
- *Existing of G invariant Hermitian form*

Hint: Averaging: $x \mapsto \frac{\sum gx}{\#G}$

Definition 4. $\mathbb{C}[X]$

Exercise 5.

- $\mathrm{Hom}_\mathbb{C}(\mathbb{C}[X], \mathbb{C}[Y]) = \mathbb{C}[X \times Y]$
- $\mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \mathbb{C}[X \times Y]^G = \mathbb{C}[X \times Y/G]$
- $\dim\mathrm{Hom}_G(\mathbb{C}[X], \mathbb{C}[Y]) = \#(X \times Y/G)$

Definition 6.

- *algebra*
- *group algebra* $\mathbb{C}[G] := \{G \to \mathbb{C}\} = span\{\delta_g\}$
  - $\delta_g * \delta_h = \delta_{gh}$
  - *action LR of* $G \times G$
    - $L(g)(\delta_h) = \delta_{gh}$
    - $R(g)(\delta_h) = \delta_{hg^{-1}}$

Exercise 7.

- $f * g(x) =?$
- *G-rep* $\Leftrightarrow$ $\mathbb{C}[G]$*-rep*
- $RL((g, h))(f)(x) =?$

Definition 8.

- *Dual representation* π^*
- *external tensor product* $(G_1, \pi_1, V_1) \boxtimes (G_2, \pi_2, V_2) = (G_1 \times G_2, \pi_1 \boxtimes \pi_2, V_1 \otimes V_2)$
- *tensor product* $(G, \pi_1, V_1) \boxtimes (G, \pi_2, V_2) = (G, \pi_1 \otimes \pi_2, V_1 \otimes V_2)$

- *$Hom_{\mathbb{C}}$ denotes both rep. of $G \times H$ and of G depending on the context.*

Exercise 9.

- $Hom_{\mathbb{C}}(V, W) \cong V^* \boxtimes W$
- $Hom_G(V, W) = Hom_{\mathbb{C}}(V, W)^G$
- $V \boxtimes W \in irr(G \times H) \iff V \in irr(G)$ & $W \in irr(H)$
- $V \boxtimes W \simeq V' \boxtimes W' \iff V \simeq V'$ & $W \simeq W'$

Definition 10.

- $a_\pi : \mathbb{C}[G] \to End_{\mathbb{C}}(\pi)$
  - $a_\pi(\delta_g) = \pi(g)$
- $m_\pi : End_{\mathbb{C}}(\pi) \to \mathbb{C}[G]$
  - $m_\pi(A)(g) = tr(A\pi(g^{-1}))$
  - $m_\pi(v \otimes \phi)(g) = \phi(\pi(g^{-1})v)$ *(check)*
- $a : \mathbb{C}[G] \to \bigoplus End_{\mathbb{C}}(\pi)$
- $m : \bigoplus End_{\mathbb{C}}(\pi) \to \mathbb{C}[G]$

Exercise 11.

- *m, a are isomorphism*
  - *m, a are morphisms of $G \times G$ representations*
  - *a is a morphism of algebras*
  - *a is injection*
  - *a_π is surjection*
  - $(\forall B, tr(AB) = 0) \Rightarrow (A = 0)$
  - *m_π is injection*
  - *m is injection*
- $\sum (dim\rho)^2 = \#G$
- $Z(\mathbb{C}[G]) = \{G//G \to \mathbb{C}\}$
- $\#irr(G) = \sum (dim\rho)^0 = \#(G//G) = \frac{\{g,h|[g,h]=1\}}{\#G}$

Definition 12. $\mathbb{C}[X]$

Exercise 13.

- *$irr(G) =?, G$ is abelian.*
  - *Any 2 commuting operators on a f.d. vec. space have a common eigenvector.*
  - *Any irreduseble representation of an abelian group is one-dimentional.*
  - *Alterative approach: Use the classification of finite abelian groups.*
- $irr(S_i) =?, i = 1, 2, 3, 4$
  - *Hint: conceder the action of S_i on the $(i-1)$-s simplex and its edges, faces, etc.*

More advance material

Character theory.

Definition 14.

- $\chi_\pi(g) = tr(\pi(g))$

Exercise 15.

- $\chi_{\pi \oplus \tau} = \chi_\pi + \chi_\tau$
- $\chi_{\pi \boxtimes \tau} = \chi_\pi \boxtimes \chi_\tau$
- $\chi_{\pi \otimes \tau} = \chi_\pi \chi_\tau$
- $\chi_{\pi^*}(g) = \chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$
- $\chi_\pi = m_{\pi^*}(Id)$
- $\chi_\pi \in Z(\mathbb{C}[G])$
- $\dim \mathrm{Hom}_G(\pi, \tau) = \langle \chi_\pi, \chi_\tau \rangle := \frac{\sum_{g \in G} \overline{\chi_\pi} \chi_\tau}{\#G}$

  - $\dim\pi^G = \langle \chi_\pi, 1\rangle$
- $\{\chi_\rho\}$ *for an orthonormal basis of* $Z(\mathbb{C}[G])$
- $a_\rho \circ m_\rho = \frac{\#G}{\dim\rho} Id$
- $\chi_\rho * \chi_\sigma = 0$ *if* $\sigma \ncong \rho$
- $\chi_\rho * \chi_\rho = \frac{\#G}{\dim\rho}\chi_\rho$
- *Compute characters of irreps of* $S_i, i = 1, 2, 3, 4$.

Theorem 16.

$$\sum_{\rho\in irr(G)} \frac{\chi_\rho(x)}{\dim\rho} = \frac{\#\{(g,h)\in G^2 | [g,h] = x\}}{\#G}$$

Exercise 17. *deduce:*

- $\sum_{\rho\in irr(G)} \frac{\chi_\rho(x)}{\dim^{2n-1}\rho} = \frac{\#\{(g_1,h_1,\dots,g_n,h_n)\in G^{2n} | [g_1,h_1]\cdots[g_n,h_n]=x\}}{\#G^{2n-1}}$
- $\sum_{\rho\in irr(G)} \frac{1}{\dim^{2n-2}\rho} = \frac{\#\{(g_1,h_1,\dots,g_n,h_n)\in G^{2n} | [g_1,h_1]\cdots[g_n,h_n]=1\}}{\#G^{2n-1}}$

how to define characters without representations.

Definition 18. *Let* A *be an algebra with a positive definite Hermitian form.*

- *An element* $a \in A$ *is positive definite iff* $\forall b \in A.\langle ab, b\rangle \geq 0$.
- *An positive definite element* $a \in A$ *is extremal iff it can not be written as a sum of 2 linearly independed positive definite elements.*
- *a minimal idempotent is an idempotent that can not be written as a sum of 2 non-zero idempotents.*

Exercise 19. *Let* $f \in Z(\mathbb{C}[G])$.

- *the following are equivalent:*
  - f *is positive definite in* $Z(\mathbb{C}[G])$
  - f *is positive definite in* $\mathbb{C}[G]$
  - f *is a positive linear combination of characters*
- *Assume that* $||f|| = 1$. *Then the following are equivalent:*
  - f *is extremal positive definite in* $Z(\mathbb{C}[G])$.
  - f *is a positive scalar multiple of a minimal idempotent*
  - f *is a character*

Frobenius formula.

Definition 20.

- $G \sim H \iff \mathbb{C}[G] \simeq \mathbb{C}[H]$
- $\zeta_G(s) = \sum \dim^{-s}\rho$

Exercise 21.

- $G \sim H \iff \dim(irr(G)) = \dim(irr(H)) \iff \zeta_G = \zeta_H \iff \zeta_G(2n-2) = \zeta_H(2n-2), \forall n \in \mathbb{N}$.
- *If* G, H *abelian and* $\#G = \#H$ *then* $G \sim H$.

Proof of theorem 16.

- Let $c_g := 1_{Ad(G)\cdot g} \in Z(\mathbb{Z}[G]) \subset Z(\mathbb{C}[G])$. Then, $a_\rho(c_g) = \frac{\#(Ad(G)\cdot g)\cdot\chi_\rho(g)}{\dim\rho} Id$.
- Let $f(x) := \frac{\#\{(g,h)\in G^2 | [g,h]=x\}}{\#G}$. Then, $f = \sum\langle f, \chi_\rho\rangle \cdot \chi_\rho$.

- $$\langle f,\chi_\rho\rangle = \frac{1}{\#G^2}\sum_{x\in G}\#\{(g,h)\in G^2|[g,h]=x\}tr(\rho(x)) = \frac{1}{\#G^2}\sum_{g,h\in G} tr(\rho([g,h])) =$$
$$= \frac{1}{\#G^2}tr\left(a_\rho\left(\sum_{g,h\in G}\delta_{ghg^{-1}h^{-1}}\right)\right) = \frac{1}{\#G^2}tr\left(a_\rho\left(\sum_{g\in G, j\in Ad(G)\cdot g^{-1}}\#(G_g)\cdot\delta_{gj}\right)\right) =$$
$$= \frac{1}{\#G^2}tr\left(a_\rho\left(\sum_{g\in G}\#(G_g)\delta_g * c_{g^{-1}}\right)\right) = \frac{1}{\#G^2}tr\left(a_\rho\left(\sum_{g\in G}\frac{\#(G_g)\cdot\#(Ad(G)\cdot g^{-1})\cdot\chi_\rho(g^{-1})}{\dim\rho}\delta_g\right)\right) =$$
$$= \frac{1}{\dim\rho\cdot\#G}tr\left(a_\rho\left(\sum_{g\in G}\chi_\rho(g^{-1})\delta_g\right)\right) = \frac{1}{\dim\rho\cdot\#G}\sum_{g\in G}\chi_\rho(g^{-1})tr(\rho(g)) = \frac{1}{\dim\rho}\langle\chi_\rho,\chi_\rho\rangle = \frac{1}{\dim\rho}$$

□

Conjecture 22.
$$SL_d(\mathbb{Z}/p^n\mathbb{Z}) \sim SL_d(\mathbb{F}_p[t]/t^nF_p[t])$$

Exercise 23. $\frac{\#G}{\dim\rho}\in\mathbb{Z}$

- $\forall a\in\mathbb{Z}[G], \exists$ *monic* $p\in\mathbb{Z}[t]$ *s.t.* $p(a)=0$.
- $\frac{\#(Ad(G)\cdot g)\cdot\chi_\rho(g)}{\dim\rho}$ *is alg. integer.*
- $\chi_\rho(g)\in\mathbb{Z}\left(\sqrt[\#G]{1}\right)$
- $\frac{\#G}{\dim\rho} = \sum_{g\in G//G}\frac{\#(Ad(G)\cdot g)\cdot\chi_\rho(g)}{\dim\rho}\overline{\chi_\rho(g)}$.
- $\frac{\#G}{\dim\rho}$ *is alg. integer.*