1 Lesson 2, 27/10/13

We want to analyze the space of distributions - $C_c^{-\infty}$. This space turns out to be not metrizable. However, we can add distributions and multiply them by a scalar, and we can also define topology without problems. This suggest using topological linear spaces.

1.1 Topological linear spaces

A *topological linear space* is a linear space with a topology, s.t. multiplication by scalar and vectors addition is continuous. This demand limits the topology we can have. For example, giving the space discrete topology will force a discrete topology on the field.
Since addition of points is continuous, translation is also continuous. This makes all the points in the space "similar" – the open sets of every point x are the same as those around 0.

We'd like our spaces to be "nice". Specifically: Hausdorff and of finite dimension. We'd also like **local convexity**: there is a basis for the topology of convex sets.

Exercise 1.1 *Find a topological linear space which is not locally convex (not necessarily of finite dimension.*

Exercise 1.2 *Let V be a locally convex linear topological space. Prove that V is Hausdorff iff $\{0\}$ is a closed set. In this case we say the space is "separated".*

Exercise 1.3 *Show that any finite dimensional Hausdorff locally convex space is isomorphic to F^n. This is also true for linear topological spaces that are not locally convex, but the proof is harder.*

We'll return to local convexity later in this class.

1.2 Defining completeness

A problem arises when defining completeness. A usual definition of completeness is by convergence of Cauchy series. Even though we don't have a metric on V, we can define Cauchy series:
A series $\{x_n\} \subset V$ is called a *Cauchy series*, if for every neighbourhood U of $0 \in V$ there is an index $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_n - x_m \in U$.
However, when our space doesn't have a metric, its sequential closure won't always coincide with the "expected" closure. Thus, when we won't have a metric on our space, we'll define completeness of a space without using sequences,[1] but using the property: **A complete space can't be embedded as a dense set in another complete space** (remember that a space is always a dense subset of its completion).

[1] Other options are definitions using filters or nets = uncountable sequences.

So:
A topological linear space is called *sequentially complete* if every Cauchy sequence in it converges. However, the space is called *complete* if for every embedding $\phi : V \to W$ which is an isomorphism $V \cong \phi(V)$, the image $\phi(V)$ is closed.

Exercise 1.4 *Find a sequentially complete space which is not complete (We'll later give a complicated example).*

A complete space $\bar{V}$ will be called a *completion* of V if there is and embedding $i : V \to \bar{V}$, where $i(V)$ is dense in $\bar{V}$ and is isomorphic to V. A different definition can be made using a **universal quality**:
An embedding $i : V \to \bar{V}$ is a *completion* of V if:
(a) $\bar{V}$ is complete.
(b) For every map $\psi : V \to W$ where W is complete, there is a unique map $\phi_W : \bar{V} \to W$, such that $\psi \equiv \phi_W \circ i$.

Exercise 1.5 **Show these two definitions of completeness are equivalent.*

Our definition of completion – using the desired property – saves us dealing with nets or filters. However, to show that such completion exists will require using them. Proving it existence was left as an exercise.

Exercise 1.6 **Show that every linear topological Hausdorff space has a completion.*

1.3 Locally convex spaces

In a locally convex space we have a basis to the topology of convex sets. We can assume all the sets are symmetric, using symmetrization ($A \mapsto conv(-A \bigcup A)$) + scaling + bounding between two symmetric sets.
However, there is a bijection between semi-norms of the space and symmetric convex sets. Given a semi-norm, we take its unit ball (it's symmetric by absolute homogeneity and convex by the triangle inequality). Given a convex symmetric set $C \subseteq V$, we'll define a semi-norm: $n_C(v) := \inf\{\alpha > 0 \mid \frac{v}{\alpha} \in C\}$.

Note: A set $C \subseteq V$ is *absorbent* $\forall x \in V\ \exists \lambda : \frac{x}{\lambda} \in C$. i.e., multiplying C by a big enough scalar can reach every point in the space. For absorbent $C \subseteq V$ we'll have $n_C(v) < \infty$ for all $v \in V$ directly from definition. Every open set is absorbent, and thus we can define our norm for all the sets in the basis.

Note2: The semi-norm we defined isn't a norm. Specifically, if C contains the subspace span$\{v\}$, we'll get $n_C(v) = 0$ (even though $v \neq 0$). However, given the basis T for our topology, we can not get $n_C(v) = 0$ for **all** the sets $C \in T$. Since in this case we'd have span$\{v\} \subseteq \bigcap_{C \in T} C$, contradicting the Hausdorff assumption (in this case every 2 points in span$\{v\}$ are in the same open set created by

$\bigcap_{C \in T} C)$.

So, in a locally convex space there is a basis to the topology using a collection of sets that defines a system of semi-norms. Some authors use this statement as the definitions of locally convex space.

Every normed space is (Hausdorff and) locally convex, since the open balls in the space are convex, and they give a basis for the topology. We also know that every normed space is metric. However, metrizability doesn't force local convexity and vice versa. We deal with metrizable locally convex space in the next section.

1.4 Fréchet spaces

Reminder: A *Banach* space is a normed space, which is complete with respect to its norm. A *Hilbert* space is a inner product space, which is complete with respect to its inner product.

Hahn-Banach theorem: Let V be a linear space over a field F, and let $f : W \to F$ be a continuous function on a closed domain $W \subseteq V$. Then we can extend f to all of V while having $\sup\{|f(x)| \mid x \in W\} = \sup\{|f(x)| \mid x \in V\}$.

Exercise 1.7 *Let V be a locally convex space, and let $f : W \to F$ be a continuous function from a closed $W \subseteq V$. Show that f can be lifted.*

The following statement is equivalent to Hahn-Banach:

For every embedding $\phi : V \to W$ which is an isomorphism $V \cong \phi(V)$, the dual map $\phi^* : V^* \to W^*$ is onto.

Definition: A *Fréchet space* is a locally convex complete metrizable space.

Exercise 1.8 *Show that for a locally convex complete space V the following conditions are equivalent:*

- *V is metrizable (the topology can be defined by a metric).*
- *V is first-countable (the first axiom of countability applies to V).*
- *There is a collection of semi-norms $\{n_i\}_{n \in \mathbb{N}}$ that defines the basis $\{B(n_i, \epsilon) \mid \epsilon \in \mathbb{R}^+, i \in \mathbb{N}\}$ for the norm over V.*

Exercise 1.9 *Let V be a locally convex metrizable space. Prove V is complete (and it's a Fréchet space) iff it's sequentially complete.*

Let V be a locally convex space. A completion of V using some semi-norm n will eliminate all the vectors in $\ker n$. When V is a Fréchet space, we can assume the sequence of norms is ascending (=the unit balls get smaller), and the topology gets finer. [2]

Fréchet spaces have several more nice qualities:
Every surjective map $\phi : V_1 \to V_2$ between Fréchet spaces is an open map (it's actually enough that V_2 is a Fréchet space and V_1 is complete).
Defining $K := ker\phi$, it can be shown that the quotient $V/_K$ is a Fréchet space, and factor ϕ to the composition $V_1 \to V/_K \to V_2$. The map $V_1/_K \to V_2$ will be an isomorphism.
In addition, every closed map $\phi : V_1 \to V_2$ ($Im\phi = \overline{Im\phi}$) between Fréchet spaces can be similarly decomposed. First by showing $Im(\phi)$ is a Fréchet space, and then decomposing $V_1 \to Im(\phi) \to V_2$. In this case, there is an isomorphism between $V_1/_K$ and $Im(\phi)$.

1.5 Sequences spaces

Reminder: l^p is the space of all sequences $\{x_n\}_{n\in\mathbb{N}}$ over a field F, such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. It is a Banach space. For $p = 2$, it is also a Hilbert space.

Let V be the space of all the sequences which decays to zero faster than any polynomial, i.e., $\forall n \in \mathbb{N}, \ \lim_{i\to\infty} x_i \cdot i^n = 0$.
One norm over such sequences can be $||\{x_i\}||_n = \sup_{i\in\mathbb{N}}\{x_i \cdot i^n\} = ||x_i \cdot i^n||_{l^\infty}$.
In that norm we can easily see that every Cauchy sequence converges. Thus, this is an example for a Fréchet space which is not a Banach space.
The dual space V^* will be $\left\{\{x_i\}_{i\in\mathbb{N}} \mid \exists n, c : \ x_i < c \cdot i^n\right\}$. This is a union of Banach spaces, as opposed to the intersection we had when defining the completion of a Fréchet space. [3] In both these cases, our space is the completion of a sense subspace - the set of all sequences with compact support.
Actually, every separable space can be composed as a sequence space. The elements of the space will correspond to infinite sequences. The sequences with compact support will correspond to the elements in the countable dense subset of the space.

1.6 Direct limits

Say we have a sequence $\{V_n\}$ of "nice" space, with embeddings $V_n \to V_{n+1}$. Their direct limit is just $V_\infty := \bigcup_{n\in\mathbb{N}} V_n$ as a linear space. The topology is defined to get a locally convex space: A **convex** subset $U \subseteq V_\infty$ is open iff for all n, $U \bigcap V_n$ is open as a subset of V_n.
This is not a Fréchet space (it is not metrizable), but it is locally convex, and therefore can be defined by semi-norms

[2] In category theory terms: the completion will be the inverse limit of the Banach spaces defined by any finite number of norms.

[3] There we had an inverse limit of Banach spaces, and here - a direct limit.

There are many connections between sequences spaces and function spaces. Continuous functions on the unit circle correspond to sequences of their Fourier coefficients. In particular, smooth functions on the unit circle, $C^\infty(S^1)$, correspond to sequences $\{x_i\}_{i\in\mathbb{N}}$ decaying faster than all polynomials. Define the norms $||f||_i = ||f^{(i)}||_{L^\infty}$ over the functions, and the semi-norms $||x_j||_i := ||x_j \cdot j^i||_{l^\infty}$ over the functions. The spaces with their i-th semi-norms aren't equivalent, but there is an equivalence in the norm we get taking the limit.

Exercise 1.10 *Prove that equivalence, thus showing these are Fréchet spaces.*

In $C^\infty(\mathbb{R})$, it's hard to define a norm (like the expected supremum norm) since the functions don't necessarily have compact support. By selecting a compact subset $U \subset \mathbb{R}$ we can define a system of semi-norms: $n_{U,k}(f) := \sup\limits_{x\in U}\{f^{(k)}(x)\}$ (these are not norms. We can have $f|_U \equiv 0$ for non-zero f). Specifically, for a sequence $U_n := [-n, n] \subset \mathbb{R}$ we'll get an ascending sequence of semi-norms. Thus, $C^\infty(\mathbb{R})$ is a Fréchet space – locally convex and first countable.
A similar argument can show $C^\infty(\mathbb{R}^n)$ is a Fréchet space, and actually also $C^\infty(M)$ for a manifold M. In these cases we'll take the supremum over all the possible derivatives.

The space $C^\infty_K(\mathbb{R})$, where K is a compact set, has the induced topology from $C^\infty(\mathbb{R})$. Taking the union of the ascending chain of Fréchet spaces $C^\infty_{[-1,1]}(\mathbb{R}) \subset C^\infty_{[-2,2]}(\mathbb{R}) \subset ...$ will give $C^\infty_K(\mathbb{R}) = \lim\limits_{\rightarrow} C^\infty_{[-n,n]}(\mathbb{R})$ as a direct limit for every such K, and therefore $C^\infty_c(\mathbb{R}) = \lim\limits_{\rightarrow} C^\infty_{[-n,n]}(\mathbb{R})$. However, this is not a Fréchet space (it's a direct limit and not an inverse limit).
So we only get a topology for $C^\infty_K(\mathbb{R})$ (and not a metric). An basic open set will be $U_{(\epsilon_n,k_n)} := \sum\limits_{n\in\mathbb{N}} \{f \in C^\infty(\mathbb{R}) \mid \operatorname{supp} f \subseteq [-n, n], f^{(k_n)} < \epsilon_n\}$
The continuity of the generalized functions will be defined with regards to this topology.