1 Lesson 1, 22/10/13

1.1 Motivation

One of the basic examples for a generalized function is "Dirac Delta function". While it's not a function, δ_t can be viewed as $\delta_t(x) := \begin{cases} \infty & x = t \\ 0 & x \neq t \end{cases}$, and somehow also having the quality $\int\limits_{-\infty}^{\infty} \delta_t(x)dx = 1$.

Here are several possible motivations to define generalized functions:

- Let's say we want to define a standard basis for the space of continuous real functions. A possible basis can be $\{f_t(x)\}_{t\in\mathbb{R}}$ where $f_t(x) := \begin{cases} 1 & x = t \\ 0 & x \neq t \end{cases}$.
  However, the functions in L^1 are actually equivalence classes of functions, up to differences in a null set, and for every $t \in \mathbb{R}$ we have $\int\limits_{-\infty}^{\infty} f_t(x)dx = 0$.
  Thus, all the functions in that basis are actually the zero function, in L^1 perspective.
  A possible solution is switching to a basis of generalized functions, and use the delta functions $\{\delta_t(x)\}_{t\in\mathbb{R}}$ as a basis.
- Sometimes the solution for a differential equation (or even just the derivative of a function) is not a function, but only a generalized function,. Using generalized functions, we can formulate solutions in such cases.
- From physics: Dirac Delta function can describe the density of a point mass, which is an infinitely small body.

1.2 Basic definitions

Let $f \in C_c^\infty(\mathbb{R})$ (i.e., f is a smooth real function, with compact support). Since by its definition $\delta_0(x) = 0$ for all $x \neq 0$, we can expect to have:

$$\int\limits_{-\infty}^{\infty} \delta_0(x) \cdot f(x)dx = \int\limits_{-\infty}^{\infty} \delta_0(x) \cdot f(0)dx = f(0) \int\limits_{-\infty}^{\infty} \delta_0(x) = f(0)$$

This rationale motivates the following definition.
Definition: A *generalized function* is a continuous linear functional $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$.

Given a real function $f \in L^1_{LOC}$ (i.e. f is locally L^1) we'll define $\xi_f : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ to be the generalized function $\xi_f(\phi) := \int\limits_{-\infty}^{\infty} f(x) \cdot \phi(x)dx$. We'll sometimes use the notation $\langle \xi, \phi \rangle$ instead of $\xi(\phi)$.

The space of generalized real functions is noted $C_c^{-\infty}(\mathbb{R})$ (and $C(\mathbb{R}) \subset L^1_{LOC} \subset C_c^{-\infty}(\mathbb{R})$).

Exercise 1.1 *Prove that there exists a function $f \in C_c^{\infty}(\mathbb{R})$ which isn't the zero function.*

We want the generalized functions to act nicely under convergence. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions converging uniformly to f, when $f \in C_c^{\infty}(\mathbb{R})$ and $f_n \in C_c^{\infty}(\mathbb{R})$ for all n. We say this sequence *converges in $C_c^{\infty}(\mathbb{R})$* if:

1. There is a compact $U \subset \mathbb{R}$ for which $\operatorname{supp}(f) \cup \bigcup\limits_{n\in\mathbb{N}} \operatorname{supp}(f_n) \subseteq U$.
2. For every order $k \in \mathbb{N}$, the derivatives $\{f_n^{(k)}\}$ converge uniformly to the derivative $f^{(k)}$.

We say the sequence $\{f_n\}_{n\in\mathbb{N}}$ *converges weakly* to f if for every $F \in C_c^{\infty}(\mathbb{R})$ we have: $\lim\limits_{n\to\infty} \int\limits_{-\infty}^{\infty} F(x) \cdot f_n(x)dx = \int\limits_{-\infty}^{\infty} F(x) \cdot f(x)dx$.

The space of generalized functions is the completion of $C_c^{\infty}(\mathbb{R})$ with respect to this weak convergence.

Exercise 1.2 *Find a sequence of functions $\{f_n\}_{n\in\mathbb{Z}}$ (when $f_n \in C_c^{\infty}(\mathbb{R})$ for all n) that weakly converges to Dirac Delta function.*

1.3 Derivatives of generalized functions

Let $f \in C_c^{\infty}(\mathbb{R})$. We defined $\xi_f(\phi) := \int\limits_{-\infty}^{\infty} f(x) \cdot \phi(x)dx$, and thus $\xi_{f'}(\phi) = \int\limits_{-\infty}^{\infty} f'(x) \cdot \phi(x)dx$. Using integration by parts we'll get $\xi_f(\phi) = f(x) \cdot \phi(x)|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} f(x) \cdot \phi'(x)dx$. However, since ϕ and f has compact support, we know that $f(x) \cdot \phi(x)|_{-\infty}^{\infty} = 0$. Thus, we'll **define** $\xi'(\phi) := -\xi(\phi')$.

For example, the derivative of $\delta_0(x)$ can be badly described as $\delta_0'(x) := \begin{cases} \infty & x \to 0^- \\ -\infty & x \to 0^+ \\ 0 & otherwise \end{cases}$.

When $\delta_0'(x)$ is applied to some $\phi \in C_c^{\infty}(\mathbb{R})$, we'll get $\delta_0'(\phi) = -\phi'(0)$.

Exercise 1.3 *Find a function $F \in L^1_{LOC}$ for which $F' = \delta$.*

1.4 The support of generalized functions

We cannot evaluate a generalized function at a point. Therefore, we cannot just define its support by $\mathrm{supp}(\xi) := \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$. However, if for some compact $U \subset \mathbb{R}$ we have $\forall f \in C_c^\infty(U),\ \xi(f) = 0$, then evidently $\xi|_U \equiv 0$ (The notation $f \in C_c^\infty(U)$ means $\mathrm{supp}(f) \subset S \subset U$ for a compact S).
As another example for a generalized function's support: it's reasonable to expect $\mathrm{supp}(\delta_t) = \{t\}$.

So, we'd like to define $\mathrm{supp}(\xi)$ to be the union over all compact $U \subset \mathbb{R}$ such that $\forall f \in C_c^\infty(U), \xi(f) = 0$. Help us do so, by solving this exercise:

Exercise* 1.4

1. *Let U_1, U_2 be open subsets of $\mathbb{R}$. Show that if $\xi|_{U_1} \equiv \xi|_{U_2} \equiv 0$ then $\xi|_{U_1 \cup U_2} \equiv 0$.*
2. *Show this also holds for any union of such compact $\{U_i\}_{i \in I}$.*

Note: The support of δ_0' is just $\{0\}$ (we didn't prove that). And yet, given some $f \in C_c^\infty(\mathbb{R})$ for which $f(0) = 0, f'(0) \neq 0$, we'll have $\xi_{\delta'}(f) \neq 0$. In other words, having f(0)=0 isn't enough to get $\langle \delta_0, f \rangle = 0$, and we'll actually need f to be zero in some neighbourhood of 0.

Exercise* 1.5 *Find all the generalized functions $\xi \in C_c^{-\infty}(\mathbb{R})$ for which* $\mathrm{supp}(\xi) = \{0\}$.

1.5 Products and convolutions of generalized functions

Let $f \in C_c^\infty(\mathbb{R}), \xi \in C_c^{-\infty}(\mathbb{R})$. We'd like to have $(f \cdot \xi)(\phi) = \int\limits_{-\infty}^{\infty} \xi(x) \cdot f(x) \cdot \phi(x) dx$. Thus, we'll define $(f \cdot \xi)(\phi) := \xi(f \cdot \phi)$.
Actually, even though we can multiply every such f and ϕ, the product of *two generalized functions* will not always be defined. Notice that indeed in some topologies the product of two Cauchy sequences isn't always a Cauchy sequence.

Recall that given two functions f, g, their *convolution* is the function $f * g(x) := \int\limits_{-\infty}^{\infty} f(t) \cdot g(x - t) dt$. The convolution of two smooth functions will always be smooth. In addition, if f, g have compact support, than so will $f * g$ (actually, $\mathrm{supp}\, f * g$ is the *Minkowski sum* of $\mathrm{supp}\, f$ and $\mathrm{supp}\, g$).

Given $f, \phi \in C_c^\infty(\mathbb{R})$ we can write $f * \phi(x) = \xi_f(\tilde{\phi_x})$, where $\tilde{\phi_x}(t) := \phi(x - t)$. This gives the motivation to define the convolution $\xi * \phi$ to be the function $\xi * \phi(x) = \xi(\tilde{\phi_x})$ (notice: the convolution between a function and a generalized function is a function – not a generalized function).

Exercise 1.6 *Show that for $\phi \in C_c^\infty(\mathbb{R})$ we get that $\xi * \phi$ is a smooth function.*

Next is the definition for convolution of two generalized functions. We won't define it for every couple of generalized functions - only for those with compact support.
For $\xi_f, \xi_g \in C_c^{-\infty}(\mathbb{R})$ we'd like to have:

$$(\xi_f * \xi_g)(\phi) = \int\limits_{x=-\infty}^{\infty} (\xi_f * \xi_g)(x) \cdot \phi(x) dx = \int\limits_{x=-\infty}^{\infty} \int\limits_{t=-\infty}^{\infty} \xi_f(t) \cdot \xi_g(x-t) \cdot \phi(x) dt dx$$

We want to express the product as an action of a general function on a (function to generalized function) convolution. Rearranging the expression we have:

$$(\xi_f * \xi_g)(\phi) = \int\limits_{t=-\infty}^{\infty} \xi_f(t) \int\limits_{x=-\infty}^{\infty} \xi_g(x-t) \cdot \phi(x) dx dt$$

In a "usual" convolution the arguments of the multiplied functions in the integral sum up to the convolution's argument (e.g., $f * g(x) := \int\limits_{-\infty}^{\infty} f(t) \cdot g(x-t) dt$, and $x = t + (x-t)$). In our case, we'll denote $\bar{\phi}(x) := \phi(-x)$, and write:

$$\int\limits_{t=-\infty}^{\infty} \xi_f(t) \int\limits_{x=-\infty}^{\infty} \xi_g(x-t) \cdot \bar{\phi}(-x) dx dt = \int\limits_{t=-\infty}^{\infty} \xi_f(t) \cdot (\xi_g * \bar{\phi})(-t) dt = \xi_f \overline{(\xi_g * \bar{\phi})}$$

So it's settled: we'll define $(\xi_f * \xi_g)(\phi) := \xi_f(\overline{(\xi_g * \bar{\phi})})$

However, some formal justification is required – via exercises.
Given a compact $U \subset \mathbb{R}$, we'll say ρ is a *cut off function of* U if $\rho|_U \equiv 0, \rho|_V \equiv 1$, when V is the complement in $\mathbb{R}$ of some neighbourhood of U (in an earlier exercise we showed such functions exist). Thus, given some $\xi \in C_c^{-\infty}(\mathbb{R})$ with $\text{supp}(\xi) \subset U$ we will have $\xi(\phi) = \xi(\delta_U \cdot \phi)$. This can be used as a definition for ξ's action when we don't have a compact support.

Exercise* 1.7 *Show that* $(\xi * \eta)' = \xi' * \eta = \xi * \eta'$.
To do that, show that $\delta * \eta = \eta$, *and that* $\delta' * \eta = \eta'$. *Then show we have associativity:* $\delta' * (\xi * \eta) = (\delta' * \xi) * \eta$.

Exercise 1.8 *In an exercise above we showed: if* $\phi \in C_c^{\infty}(\mathbb{R})$ *then the convolution* $\xi * \phi$ *is smooth. Show that if* ϕ *is smooth, and* $\text{supp}(\xi)$ *is compact, then* $\xi * \phi$ *will still be smooth.*

1.6 Generalized functions and differential operators

A differential equation can be described by the equality "$Af = g$", where A is a differential operator.

Let's try to solve such an equation, when we assume A is a linear differential operator, and is invariant under translations (i.e., we'll have $A\bar{f} = \overline{Af}$, where $\bar{\phi}$ is any fixed translation of ϕ). An example for such operator is a differential operators with fixed coefficients (e.g., $Af := f'' + 5f' + 6f$).

A simple case is finding G for which the equation $AG = \delta_0$ holds. Given such G, and using A's invariance under translations, we get that $AG_x = \delta_x$, for $G_x(t) := G(t - x)$.
From that we get somehow that $A(f * h) = (Af) * h$ holds for any two functions f, h. Specifically: $A(G * g) = AG * g = \delta_0 * g = g$.
And so, we can find a general solution f for $Af = g$ by solving only one simpler case $AG = \delta_0$. The solution G is called *Green's function of the operator.*

Exercise 1.9 *Let A be a differential operator with fixed coefficients. Choose any solution for the equation $AG = \delta_0$, and describe the conditions G have to meet without using generalized functions.*

Exercise 1.10 *Without using generalized functions, please explain the equation $A(G * g) = g$ we got for the solution G.*

Exercise 1.11 *Solve the equation $\Delta f = \delta_0$ (where Δ is the Laplacian operator).*

1.7 Regularization of generalized functions

Let $\{\xi_\lambda\}_{\lambda \in \mathbb{C}}$ be a family of generalized functions. We say the family is *analytic* if $\langle \xi_\lambda, f \rangle$ is analytic for every $f \in C_c^\infty(\mathbb{R})$.

Example: We denote $x_+^\lambda := \begin{cases} x^\lambda & x > 0 \\ 0 & x \leq 0 \end{cases}$, and define the family by $\xi_\lambda := x_+^\lambda$.
The behaviour of the function changes as λ changes: When $Re(\lambda) > 0$ we'll have a nice continuous function; If $Re(\lambda) = 0$ We'll get a step function; And for $Re(\lambda) \in (-1, 0)$, x_+^λ will not be bounded. We'd like to extend the definition analytically for $Re(\lambda) < -1$.
A derivation of x_+^λ (both as a complex function or as defined for a generalized function) gives $\xi'_\lambda = \lambda \cdot \xi_{\lambda-1}$. This is a functional equation, that enables us to **define** $\xi_{\lambda-1} := \frac{\xi'_\lambda}{\lambda}$, and thus extend ξ_λ to every $\lambda \in \mathbb{C}$. This extension isn't analytic, but is meromorphic: it has a pole in $\lambda = 0$, and by the extension formula, in $\lambda = -1, -2, ...$

This is an example for *a meromorphic family of generalized functions*. Let's give a formal definition. Our $\{\xi_\lambda\}_{\lambda \in \mathbb{C}}$ has a set of poles $\{\lambda_n\}$ (poles are always discrete), whose respective orders are denoted $\{d_n\}$. The family will be called *meromorphic* if every pole λ_i has a neighbourhood U_i, such that $\langle \xi_\lambda, f \rangle$ is analytic for every $f \in C_c^\infty(\mathbb{R})$ and $\lambda \in U_i$.

Exercise 1.12 *For the above example $\xi_\lambda := x_+^\lambda$, find the order and the leading coefficient for every pole.*

Here's the general technique we'll use:
We write $\langle \xi_\lambda, f\rangle = \int\limits_0^1 x^\lambda \cdot f(x)dx + \int\limits_1^\infty x^\lambda \cdot f(x)dx$. The second integral converges.
In order to evaluate to first, we'll express the Taylor series of f up to the N-th order:

$$f = \sum_{i=0}^{N} \frac{f^{(i)}(x)}{i!} \cdot x^i + g(x)$$

Where g(x) is the remainder.
When $Re(\lambda)$ is big enough, we can integrate f term-by-term, and get:

$$\int_0^1 x^\lambda \cdot f(x)dx = \int_0^1 x^\lambda \cdot g(x)dx + \sum_{i=0}^{N} a_i \int_0^1 x^{\lambda+i}dx = \int_0^1 x^\lambda \cdot g(x)dx + \sum_{i=0}^{N} a_i \cdot \left.\frac{x^{\lambda+i+1}}{\lambda+i+1}\right|_0^1$$

Thus, we'll define to be $\int\limits_0^1 x^{\lambda+i}dx = \frac{1}{\lambda+i+1}$ everywhere. This will give us an expression for $\langle \xi_\lambda, f\rangle$ for every value of λ, which will coincide with with the analytic continuation of ξ_λ.

Another example: For a given $p \in \mathbb{C}[x_1, ...x_n]$, we denote similarly $p_+(x_1, ...x_n)^\lambda := \begin{cases} p(x_1, ...x_n)^\lambda & x > 0 \\ 0 & x \leq 0 \end{cases}$. The problem of finding the meromorphic continuation for a general polynomial was open for a while. It was solved by (Bernstein by) defining a differential operator $Dp_+^\lambda := b(\lambda) \cdot p_+^{\lambda-1}$, where $b(\lambda)$ was a polynomial pointing on the location of the poles.

Exercise 1.13 *Solve the problem of finding an analytic continuation for* $p_+(x_1, ...x_n)^\lambda$ *in the case* $p(x, y, z) := x^2 + y^2 + z^2 - a$.

Exercise* 1.14 *Solve the problem of finding an analytic continuation for* $p_+(x_1, ...x_n)^\lambda$ *in the case* $p(x, y, z) := x^2 + y^2 - z^2$.

2 Lesson 2, 27/10/13

We want to analyze the space of distributions. This space turns out to be not metrizable. However, we can add distributions and multiply them by a scalar, and we can also define topology without problems. This suggest using topological linear spaces.

2.1 Topological linear spaces

A *topological linear space* is a linear space with a topology, s.t. multiplication by scalar and vectors addition is continuous. This demand limits the topology we can have. For example, giving the space discrete topology will force a discrete topology on the field.
Since addition of points is continuous, translation is also continuous. This makes all the points in the space "similar" – the open sets of every point x are the same as those around 0.

We'd like our spaces to be "nice". Specifically: Hausdorff and of finite dimension. We'd also like **local convexity**: there is a basis for the topology of convex sets.

Exercise 2.1 *Find a topological linear space which is not locally convex (not necessarily of finite dimension.*

Exercise 2.2 *Let V be a locally convex linear topological space. Prove that V is Hausdorff iff $\{0\}$ is a closed set. In this case we say the space is "separated".*

Exercise 2.3 *Show that any finite dimensional Hausdorff locally convex space is isomorphic to F^n. This is also true for linear topological spaces that are not locally convex, but the proof is harder.*

We'll return to local convexity later in this class.

2.2 Defining completeness

A problem arises when defining completeness. A usual definition of completeness is by convergence of Cauchy series. Even though we don't have a metric on V, we can define Cauchy series:
A series $\{x_n\} \subset V$ is called a *Cauchy series*, if for every neighbourhood U of $0 \in V$ there is an index $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_n - x_m \in U$.
However, when our space doesn't have a metric, its sequential closure won't always coincide with the "expected" closure. Thus, when we won't have a metric on our space, we'll define completeness of a space without using sequences,[1] but using the property: **A complete space can't be embedded as a dense set in another complete space** (remember that a space is always a dense subset of its completion).

[1] Other options are definitions using filters or nets = uncountable sequences.

So:
A topological linear space is called *sequentially complete* if every Cauchy sequence in it converges. However, the space is called *complete* if for every embedding $\phi : V \to W$ which is an isomorphism $V \cong \phi(V)$, the image $\phi(V)$ is closed.

Exercise 2.4 *Find a sequentially complete space which is not complete (We'll later give a complicated example).*

A space $\bar{V}$ will be called a *completion* of V if there is and embedding $i : V \to \bar{V}$, where $i(V)$ is dense in $\bar{V}$ and is isomorphic to V. A different definition can be made using a **universal quality**:
An embedding $i : V \to \bar{V}$ is a *completion* of V if:
(a) $\bar{V}$ is complete.
(b) For every map $\psi : V \to W$ where W is complete, there is a unique map $\phi_W : \bar{V} \to W$, such that $\psi \equiv \phi_W \circ i$.

Exercise* 2.5 *Show these two definitions of completeness are equivalent.*

Our definition of completion – using the desired property – saves us dealing with nets or filters. However, to show that such completion exists will require using them. Proving it existence was left as an exercise.

Exercise* 2.6 *Show that every linear topological Hausdorff space has a completion.*

2.3 Locally convex spaces

In a locally convex space we have a basis to the topology of convex sets. We can assume all the sets are symmetric, using symmetrization ($A \mapsto conv(-A \bigcup A)$) + scaling + bounding between two symmetric sets.
However, there is a bijection between semi-norms of the space and symmetric convex sets. Given a semi-norm, we take its unit ball (it's symmetric by absolute homogeneity and convex by the triangle inequality). Given a convex symmetric set $C \subseteq V$, we'll define a semi-norm: $n_C(v) := \inf\{\alpha > 0 \mid \frac{v}{\alpha} \in C\}$.

Note: A set $C \subseteq V$ is *absorbent* $\forall x \in V\ \exists \lambda : \frac{x}{\lambda} \in C$. i.e., multiplying C by a big enough scalar can reach every point in the space. For absorbent $C \subseteq V$ we'll have $n_C(v) < \infty$ for all $v \in V$ directly from definition. Every open set is absorbent, and thus we can define our norm for all the sets in the basis.

Note2: The semi-norm we defined isn't a norm. Specifically, if C contains the subspace span$\{v\}$, we'll get $n_C(v) = 0$ (even though $v \neq 0$). However, given the basis T for our topology, we can not get $n_C(v) = 0$ for **all** the sets $C \in T$. Since in this case we'd have span$\{v\} \subseteq \bigcap\limits_{C \in T} C$, contradicting the Hausdorff assumption.

So, in a locally convex space there is a basis to the topology using a collection of sets that defines a system of semi-norms. Some authors use this statement as the definitions of locally convex space.
Every normed space is (Hausdorff and) locally convex, since the open balls in the space are convex, and they give a basis for the topology. We also know that every normed space is metric. However, metrizability doesn't force local convexity and vice versa. We deal with metrizable locally convex space in the next section.

2.4 Fréchet spaces

Reminder: A *Banach* space is a normed space, which is complete with respect to its norm. A *Hilbert* space is a inner product space, which is complete with respect to its inner product.

Hahn-Banach theorem: Let V be a linear space over a field F, and let $f : W \to F$ be a continuous function on a closed domain $W \subseteq V$. Then we can extend f to all of V while having $\sup\{|f(x)| \mid x \in W\} = \sup\{|f(x)| \mid x \in V\}$.

Exercise 2.7 *Let V be a locally convex space, and let $f : W \to F$ be a continuous function from a closed $W \subseteq V$. Show that f can be lifted.*

The following statement is equivalent to Hahn-Banach:
For every embedding $\phi : V \to W$ which is an isomorphism $V \cong \phi(V)$, the dual map $\phi^* : V^* \to W^*$ is onto.

Definition: A *Fréchet space* is a locally convex complete metrizable space.

Exercise 2.8 *Show that for a locally convex complete space V the following conditions are equivalent:*

- *V is metrizable.*
- *V is first-countable.*
- *There is a collection of semi-norms $\{n_i\}_{n\in\mathbb{N}}$ that defines the basis $\{B(n_i, \epsilon) \mid \epsilon \in \mathbb{R}^+, i \in \mathbb{N}\}$ for the norm over V.*

Exercise 2.9 *Let V be a locally convex metrizable space. Prove V is complete (and it's a Fréchet space) iff it's sequentially complete.*

Let V be a locally convex space. A completion of V using some semi-norm n will eliminate all the vectors in $\ker n$. When V is a Fréchet space, we can assume the sequence of norms is ascending (=the unit balls get smaller), and the topology gets finer. [2]

[2] In category theory terms: the completion will be the inverse limit of the Banach spaces defined by any finite number of norms.

Fréchet spaces have several more nice qualities:
Every surjective map $\phi : V_1 \to V_2$ between Fréchet spaces is an open map (it's actually enough that V_2 is a Fréchet space and V_1 is complete).
Defining $K := ker\phi$, it can be shown that the quotient $V/_K$ is a Fréchet space, and factor ϕ to the composition $V \to V/_K \to V_2$. The map $V/_K \to V_2$ will be an isomorphism.
In addition, every closed map $\phi : V_1 \to V_2$ between Fréchet spaces can be similarly decomposed. First by showing $Im(\phi)$ is a Fréchet space, and then decomposing $V_1 \to Im(\phi) \to V_2$.

2.5 Sequences spaces

Reminder: l^p is the space of all sequences $\{x_n\}_{n\in\mathbb{N}}$ over a field F, such that $\sum\limits_{n=1}^{\infty} |x_n|^p < \infty$. It is a Banach space. For $p = 2$, it is also a Hilbert space.

Let V be the space of all the sequences which decays to zero faster than any polynomial, i.e., $\forall n \in \mathbb{N}$, $\lim\limits_{i\to\infty} x_i \cdot i^n = 0$.
One norm over such sequences can be $||\{x_i\}||_n = \sup\limits_{i\in\mathbb{N}}\{x_i \cdot i^n\} = ||x_i \cdot i^n||_{l^\infty}$.
In that norm we can easily see that every Cauchy sequence converges. Thus, this is an example for a Fréchet space which is not a Banach space.
The dual space V^* will be $\left\{\{x_i\}_{i\in\mathbb{N}} \mid \exists n, c: \ x_i < c \cdot i^n\right\}$. This is a union of Banach spaces, as opposed to the intersection we had when defining the completion of a Fréchet space. [3] In both these cases, our space is the completion of a sense subspace - the set of all sequences with compact support.
Actually, every separable space can be composed as a sequence space. The elements of the space will correspond to infinite sequences. The sequences with compact support will correspond to the elements in the countable dense subset of the space.

2.6 Direct limits

Say we have a sequence $\{V_n\}$ of "nice" space, with embeddings $V_n \to V_{n+1}$. Their direct limit is just $V_\infty := \bigcup\limits_{n\in\mathbb{N}} V_n$ as a linear space. The topology is defined to get a locally convex space: A **convex** subset $U \subseteq V_\infty$ is open iff for all n, $U \bigcap V_n$ is open as a subset of V_n.

There are many connections between sequences spaces and function spaces.
Continuous functions on the unit circle correspond to sequences of their Fourier coefficients. In particular, smooth functions on the unit circle, $C^\infty(S^1)$, correspond to sequences $\{x_i\}_{i\in\mathbb{N}}$ decaying faster than all polynomials. Define the norms $||f||_i = ||f^{(i)}||_{L^\infty}$ over the functions, and the semi-norms $||x_j||_i :=$

[3] There we had an inverse limit of Banach spaces, and here - a direct limit.

$||x_j \cdot j^i||_{l^\infty}$ over the functions. The spaces with their i-th semi-norms aren't equivalent, but there is an equivalence in the norm we get taking the limit.

Exercise 2.10 *Prove that equivalence, thus showing these are Fréchet spaces.*

In $C^\infty(\mathbb{R})$, it's hard to define a norm (like the expected supremum norm) since the functions don't necessarily have compact support. By selecting a compact subset $U \subset \mathbb{R}$ we can define a system of semi-norms: $n_{U,k}(f) := \sup_{x \in U}\{f^{(k)}(x)\}$ (these are not norms. We can have $f|_U \equiv 0$ for non-zero f). Specifically, for a sequence $U_n := [-n, n] \subset \mathbb{R}$ we'll get an ascending sequence of semi-norms. Thus, $C^\infty(\mathbb{R})$ is a Fréchet space – locally convex and first countable.
A similar argument can show $C^\infty(\mathbb{R}^n)$ is a Fréchet space, and actually also $C^\infty(M)$ for a manifold M. In these cases we'll take the supremum over all the possible derivatives.

The space $C_K^\infty(\mathbb{R})$ has the induced topology from $C^\infty(\mathbb{R})$. Taking the union of the ascending chain $C_{[-1,1]}^\infty(\mathbb{R}) \subset C_{[-2,2]}^\infty(\mathbb{R}) \subset \ldots$ will give $C_K^\infty(\mathbb{R}) = \varinjlim C_{[-n,n]}^\infty(\mathbb{R})$ as a direct limit. However, this is not a Fréchet space (it's a direct limit and not an inverse limit).
So we only get a topology for $C_K^\infty(\mathbb{R})$ (and not a metric). An basic open set will be $U_{(\epsilon_n,k_n)} := \sum_{n \in \mathbb{N}} \{f \in C^\infty(\mathbb{R}) \mid \operatorname{supp} f \subseteq [-n, n], f^{(k_n)} < \epsilon_n\}$
The continuity of the generalized functions will be defined with regards to this topology.

3 Lesson 6, 27/11/13

3.1 leftover from last lesson: p-adic expansions

Given $q \in \mathbb{Q}$ – let's write its p-adic expansion. If $q \in \mathbb{Z}$, that's just writing its p-base expansion.
Let $x := \frac{m}{n}$ be some rational number, with $(n, m) = 1$. We'll describe the expansion when $p \nmid m$ (that's when $x \in \mathbb{Z}_p \cap \mathbb{Q}$). In this case the will be no digits to the right of the (p-adic) point. On the general case we can divide x by p^k for some k, and move the point accordingly.
We can't take remainder of x modulo p, as with integers. Instead, we can calculate the fraction $x\frac{m}{n}$ in $\mathbb{F}_{p^n}$ for $n \in \mathbb{N}$. Thus, the expansions of in $\mathbb{Q}_p$ is calculated inductively:

- Write the digit $x_0 := [\frac{m}{n}] \in \mathbb{F}_p$.
- The nominator of the difference $\frac{m}{n} - x_0 = \frac{m-n\cdot x_0}{n}$ is divisible by p. Redefine our fraction to be $x := \frac{1}{p} \cdot (\frac{m}{n} - x_0)$, and continue inductively.

Example: The expansion of $x := \frac{1}{3}$ in $\mathbb{Q}_5$. The first digit will be $x_0 := [\frac{1}{3}] = 2 \in \mathbb{F}_p$. Next, we calculate $\frac{1}{5} \cdot (\frac{1}{3} - 2) = \frac{-1}{3}$. In $\mathbb{F}_5$ the fraction is $\frac{-1}{3} = 3$, and that's the digit is $x_1 = 3$. The next stages are:

$\frac{1}{5} \cdot (\frac{1}{3} - 3) = \frac{-2}{3} \implies x_2 = 1$,
$\frac{1}{5} \cdot (\frac{-2}{3} - 1) = \frac{-1}{3} \implies x_3 = 3$.
We get: $(\frac{1}{3})_5 = ...13132.0$.

The result is equivalent to $(\frac{1}{3})_5 = 2 + 3 \cdot 5 + 1 \cdot 25 + 3 \cdot 125 + ...$
Indeed, $2 \equiv \frac{1}{3}$ in $\mathbb{Z}/_{5\mathbb{Z}}$, $2 + 3 \cdot 5 = 17 \equiv \frac{1}{3}$ in $\mathbb{Z}/_{25\mathbb{Z}}$, $2 + 3 \cdot 5 + 1 \cdot 25 = 42 \equiv \frac{1}{3}$ in $\mathbb{Z}/_{125\mathbb{Z}}$, etc.

3.2 Distributions over the p-adics

Let X be a l-space. We want to define smooth functions on this space, $f \in C^\infty(X)$. We don't have a notion of derivatives (even though X is a complete space). For example, for $X = \mathbb{Q}_p$ the usual derivative is $\lim\limits_{t \to 0} \frac{f(x+t) - f(x)}{t}$ – that's a quotient of a complex number by an element in $X = \mathbb{Q}_p$, which we didn't define.
We'll use a different approach. A function f from X to the field will be called a *smooth function* if for every point $x \in X$ there is an open neighbourhood U such that the restriction $f|_U$ is constant.

Exercise 3.1 *Let X be an l-space. Show that the smooth functions $C^\infty(X)$ separates the points in X.*

Assuming this exercise, the Stone-Weierstrass theorem implies that $C^\infty(X)$ is dense in $C(X)$. We'll denote $C_C^\infty(X) \subset C^\infty(X)$ the set of smooth functions of compact support. These functions are called *Schwartz functions*, and we'll also denote them $S(X)$.
Remark: Conversely, in $\mathbb{R}^n$, the Schwartz functions are the functions whose derivatives decrease faster than every polynomial, and $C_C^\infty(\mathbb{R}^n) \subset S(X) \subset C^\infty(\mathbb{R}^n)$. We'll define them in the next lectures.

We'll define the dual space $S^*(X)$ to be all the linear functions $f : S(X) \to F$. Usually we'd also demand the functions to be continuous, but here we don't, and adding that demand won't change a lot.[4] An intuition is that $S(X)$ can be defined as a direct limit – and its induced topology as such will be "every subset in $S(X)$ is open". Thus, every function in $S^*(X)$ will be continuous, and adding that demand won't change anything.
As oppose to $S(X)$, there is a reasonable topology over $S^*(X)$. That is the weak topology by which $f = \lim\limits_{n\to\infty} f_n$ if $f(\phi) = \lim\limits_{n\to\infty} f_n(\phi)$ for all $\phi \in S(X)$. The space $S^*(X)$ is complete with respect to this topology.

[4] on l-spaces the schwartz functions can be defined as the limit of finite unions over compact sets. Over $\mathbb{R}$, on the other hand, $S(X)$ is defined as an inverse limit – the limit of finite intersections

3.3 Distributions supported on a subspace

Let X be a linear space, and let $Z \subset X$ be some closed subspace. Over $\mathbb{R}$ we tried to analyze the distributions on X that are supported on Z (and got some filtration on X).

Exercise 3.2 *Prove that $S^*(X)$ over an l-space is a sheaf. In particular, you need to prove the partition of unity over local fields. It is easier to prove since you can refine all open covers.*

We say $f \in S^*(X)$ is *supported* on $Z \subset X$, and denote $f \in S^*_Z(X)$, if the restriction $f|_S(Z)$ isn't zero. This restriction gave us an inclusion $i : S^*(Z) \to S^*_Z(X)$ over $\mathbb{R}$.

Exercise 3.3 *Show that i is an inclusion when X is an l-space. Prove that by showing the dual map $S(X) \to S(Z)$ is onto. The idea is similar to Tietze extension theorem.*

The map i wasn't onto over $\mathbb{R}$. For example, for $Z := \{0\} \subset \mathbb{R}$, the derivatives $\delta_0^{(n)}$ were in $S^*_Z(\mathbb{R}^n)$ but not in the image of i.
Claim: Let X be an l-space, and $Z \subset X$ a closed subspace. Then the inclusion $i : S^*(Z) \to S^*_Z(X)$ is also onto.
Proof: Let $\xi \in S^*(X)$ be a distribution with $\operatorname{supp} \xi \subset Z$.
Over $\mathbb{R}$ we looked at the quotient $S(Z)/_{S(X\backslash Z)}$. Here, need to show there is a one-to-one map from the quotient to $S(Z)$. In other words, that the kernel of the inclusion $S(Z) \to S(X)$ are the Schwartz functions over $X\backslash Z$. This follows from the Schwartz functions being locally constant, and so every f zeroing on Z also zeroes on a neighbourhood of Z.
This gave us an exact sequence $0 \to S^*(Z) \to S^*(X) \to S^*(X\backslash Z) \to 0$.[5] In $\mathbb{R}$ we had the sequence $0 \to S^*_Z(X) \to S^*(X) \to S^*(X\backslash Z)$.

Exercise 3.4 *(easy) Let V be a vector space (maybe infinite-dimensional) over a field K, and $L \subset V$ a linear subspace. Show that $\forall f \in L^* \ \exists g \in V^* : g|_L \equiv f$. Use Zorn's lemma.*

So far we showed two advantages of distributions on l-spaces over distributions on $\mathbb{R}^n$:

1. Every distribution ξ supported on some $Z \subset X$ is also supported on a neighbourhood of Z.

2. The map $i : S^*(Z) \to S^*_Z(X)$ is onto.

Both these qualities can be achieved over $\mathbb{R}^n$ by switching from $C^\infty_c(\mathbb{R})$ to real-valued Schwartz functions. A third advantage is:

[5]we didn't show that $S(X\backslash Z) \subset S(X)$, but that's immediate without a topology on the space.

Exercise 3.5 *Let X, Y be l-spaces. Given $f_1 \in S(X), f_2 \in S(Y)$, consider the bilinear map $\phi : S(X) \otimes S(Y) \to S(X \times Y)$ where $(\phi(f))(x, y) := f_1(x) \cdot f_2(y)$. Show that ϕ is locally constant and an isomorphism of vector spaces (the injective part might be hard).*

Exercise 3.6 *The previous exercise gives a map $S^*(X) \otimes S^*(Y) \to S^*(X \times Y)$. Show it's dense and one-to-one (again, might be hard to show injectivity).*

Next, we'll denote $S(X, V)$ for l-spaces as the set of Schwartz functions with values in V (like we did over $\mathbb{R}$).

Exercise 3.7 *Show that we still have $S(X, V) \cong S(X) \otimes V$ and $S^*(X, V) \cong S^*(X) \otimes V^*$ when V is finite-dimensional. There first isomorphism should be easier than the proof over $\mathbb{R}$, but the second should be similar.*

From this exercise we'll get the isomorphisms $S(X, S(Y)) \cong S(X \times Y)$ and $S^*(X, S(Y)) \cong S^*(X \times Y)$. This will help us reduce questions on distributions on products $X \times Y$ to (albeit more complicated) distributions on X.

3.4 Smooth measures

Let F be a local field with $charF \neq 0$, and let V be a finite-dimensional linear space over F. We'll denote h(v) the (one-dimensional) space of Haar measures over V, and $\Omega^n(V)$ the multilinear n-forms over it.[6] The density space is again $Dens(V) := |\Omega^n(V)|$, where $|\cdot|$ is defined by $|L| := \{f : L^* \to \mathbb{R} \mid f(kv) = |k| \cdot f(v)\}$.
Over the p-adics, the absolute value is the "normalized" one, which we defined as $|k| := \frac{\mu(kA)}{\mu(A)}$. We don't have the orientations space $ori(V)$ since we don't have sign function over the p-adics.

Exercise 3.8 *Show that we still have $h(V) \cong Dens(V)$.*

Remark for the exercise: Over $\mathbb{R}^n$ we had a standard measure. Over our field the standard measure is Lebesgue-like, with $\mu([0, 1]) = 1$. We extend it to V using the product measure $\mu(X \times Y) := \mu(X) \times \mu(Y)$.
The exercise deals with the effect of an endomorphism $\phi : V \to V$ on the the measure. Over $\mathbb{R}^n$ the measure was multiplied by the determinant of the Jacobian. Here we'll have a similar claim, and it's enough to prove it over the diagonal matrices and over the upper-triangular matrices.

Next, we define *smooth measures*. As over $\mathbb{R}$, these will be $S(V, h(V)) \cong S(V) \otimes (V)$.

Exercise 3.9 *Show that ξ is a smooth measure iff $\operatorname{supp} \xi$ is compact, and there exists an open compact subgroup $K \leq V$, such that $K.\xi = \xi$. Notice the difference: the Haar measures are invariant under all translations, and here we only demand invariance under translations by vectors in K.*

[6] This time $h(v) \subset S^*(V)$ is defined over "smooth" functions.

That's another advantage of distributions over local fields: we have a easy condition by which to characterize smooth measures. Later we'll define generalized functions over l-spaces.

3.5 Geometry of manifolds

Let X be a topological linear space. Given an open subset $U \subset X$, a collection of functions $O(U) \subset C(U, \mathbb{R})$ is called a *sheaf* of functions if the following conditions hold:

1. $O(U)$ is an algebra with unity.
2. For every open cover $U = \bigcup\limits_{i \in I} U_i$:
   (a) $\forall f \in O(U)$: if $\forall i \in I\ f|_{U_i} \equiv 0$ then $f \equiv 0$.
   (b) If exists a set of functions $\{f_i \in O(U_i) \mid i \in J\}$ s.t. $\forall i, j \in J$: $f_i|_{(U_i \cap U_j)} \equiv f_j|_{(U_i \cap U_j)}$ then there exists $f \in O(U)$ s.t. $\forall i \in J, f|_{U_i} \equiv f_i$.

A sheaf of continuous functions can be defined simpler by demanding: $\forall f \in C(U, \mathbb{R}),\ f|_{U_i} \in O(U_i) \implies f \in O(U)$.

The sheaf is a structure over elements with a common local attribute.
Examples for function sheaves can be continuous functions on X, smooth functions on X and smooth constant functions on it. The last one is a sheaf over $\mathbb{R}$ but not over $\mathbb{Q}_p$. Over both, however, the locally constant functions compose a sheaf. A counter-example is the collection of functions with local support. That is a global attribute of a function and as such can't define a sheaf.

To define manifolds we'll use *spaces with function sheaves*. These are pairs (X, F), with $F = O(X)$. A map $f : (X, F) \to (Y, G)$ will be a continuous map $f : X \to Y$ and its dual maps $f^* : C(U) \to C(f^{-1}(U)$ (where $f^*(\phi) := \phi \circ f$, and $U \subset Y$ is some open subset), such that $f^*(O(U)) \subseteq O(f^{-1}(U))$.
Example: for any open $U \subseteq \mathbb{R}$ the pair $(U, C^\infty(U))$ is a space with functions.

Now we can define:
A *topological manifold* is a topological Hausdorff paracompact[7] that "locally looks like $\mathbb{R}^n$. i.e., every point $x \in X$ has a open neighbourhood U and two diffeomorphisms $\psi : U \to \mathbb{R}^n, \phi : \mathbb{R}^n \to U$, s.t. $\phi \circ \psi \equiv id|_U$ and $\psi \circ \phi \equiv id|_{\mathbb{R}^n}$.

A *smooth manifold* is a space with functions $(X, C^\infty(X))$, where X is a topological manifold and for every point $x \in X$ there is a open neighbourhood U and two diffeomorphisms $\psi : (U, C^\infty(U)) \to (\mathbb{R}^n, C^\infty(\mathbb{R})), \phi : (\mathbb{R}^n, C^\infty(\mathbb{R})) \to (U, C^\infty(U))$, s.t. $\phi \circ \psi \equiv id|_U$ and $\psi \circ \phi \equiv id|_{\mathbb{R}^n}$.

[7] We need the paracompactness to assume the partition of unity holds over the manifold.

Remark: The usual definition of manifolds adds an "atlas" to the structure of X: an open cover $X = \bigcup_{i \in I} U_i$ with diffeomorphism $\phi_i : U_i \to \mathbb{R}^n$. Instead of diffeomorphisms ψ, ϕ between $(U, C^\infty(U))$ and $(\mathbb{R}^n, C^\infty(\mathbb{R}))$, it has the demand that $\phi_i \circ \phi_j^{-1}$ is differentiable.

Defining submanifolds is a bit delicate for a general space. A *submanifold* in the Euclidean space is a subset $M \subseteq \mathbb{R}^n$ for which: For every $p \in M$ there is an open neighbourhood $U_p \in \mathbb{R}^n$ s.t. $U_p \cap \mathbb{R}^n$ is the zero set of a differentiable $f : U_p \to \mathbb{R}^k$, and f has a differential of full rank (i.e., $rank(\frac{\partial f_i}{\partial x_j}(p)) = k$).
The full rank condition enables usage of the implicit function theorem. The "usual definition" of a submanifold $M \subseteq N$ asks the atlas of N to be the restriction of the atlas of M.
Example: S^n is a smooth manifold, and it's a submanifold of $\mathbb{R}^n$.

A theorem by Whitney shows that every n-dimensional manifold can be embedded in $\mathbb{R}^{2n+1}$.

Exercise 3.10 *Give an example for a set that is "almost" a smooth manifold, expect that it's not Hausdorff.*

Exercise* 3.11 *Give an example for a set that is "almost" a smooth manifold, expect that it's not paracompact. The set should be connected and one-dimensional.*

3.6 Vector bundles

A *Vector bundle* is a smooth family of vector spaces. Given two (smooth) manifolds M, E and a map $p : E \to M$, we'd like to define E as a family of vector spaces over M – a vector space for every fiber in $\{p^{-1}(x) \mid x \in M\}$. For that end, we demand that for every point $x \in M$ exist an open neighbourhood U and a diffeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{R}^n$ that factors the diffeomorphism $p^{-1}(U) \to M$. In addition, we'll demand the restriction of ϕ to each fiber of p is an isomorphism of vector spaces $\phi|_{p^{-1}(x)} : p^{-1}(x) \to \mathbb{R}^n$.
Example: $E := M \times \mathbb{R}^n$ with the constant (smooth) bundle.
Example: The *Mobius strip* is homeomorphic to $I \times S^1$. By extending each segment I to $\mathbb{R}$, we can define a bundle over the manifold S^1. This way, the points in E are pairs (θ, x), where x runs over the points of the line of angle $0.5 \cdot \theta$.

Exercise 3.12 *Define rigourously the vector bundle in the 2nd example, and show it's not diffeomorphic to the bundle $S^1 \times \mathbb{R}$. You can assume the Mobius strip isn't diffeomorphic to the S^1.*

4 Lesson 7, 2/12/13

4.1 Tangent space of a manifold

The tangent space of a manifold M in a point x should describe the directions in which we can "walk" on the manifold from x. This space will be a the complement to the normal space at x – the linear subspace of normals to x. The dimension of the tangent space will be equal to the dimension of M for "most of" the points $x \in M$. In "problematic" points, such as corners, the dimension will decrease.
We define the tangent space now formally, first for a linear space.
Let V be a linear space, $U \subseteq V$ an open set and $x \in V$ (especially, $dimU = dimV$). We define The *tangent space in* x will be defined $T_xU := V$. Given a smooth function $\phi : U \to U'$, the *differential* $D\phi : T_xU \to T_{\phi(x)}U'$ is the induced map on the tangent spaces. The differential is defined as the 1st-order approximation of ϕ. Namely, given any smooth $\psi : V \to W$, we take a map $\psi_1 : V \to W$ for which $\lim\limits_{y \to x} \frac{\psi(y) - \psi_1(y)}{\|y-x\|} = 0$, and the differential $D\psi$ will be defined by $\psi(y) = \psi(x) + D\psi(y - x)$, for every $y \in U$.

Now, let's define a tangent space to a general smooth manifold M.
There are several equivalent definition:

1. $V \in T_x(M)$ is a correspondence $v_\phi \in T_0(V)$ for any map $\phi : (V, 0) \to (M, x)$ s.t. given $\psi : (U, 0) \to (U', 0), \phi' : (U', 0) \to (M, x)$ we'll have $D\psi(V_\phi) \equiv V_{\phi'}$.

2. $T_x(M) := \{\phi : (R, 0) \to (M, x) \mid \phi \in C^\infty(\mathbb{R})\}$ modulo the relation $\gamma_1 \sim \gamma_2$ iff exists a neighbourhood U of x and a isomorphism $\phi : U \to \mathbb{R}^n$ s.t. $\lim\limits_{x \to 0} \frac{(\phi \circ \gamma_1)(x) - (\phi \circ \gamma_2)(x)}{x} = 0$. One should prove that if U, ϕ exists than for every other U we'll have a ϕ.

   Exercise 4.1 *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$. Show that ϕ is smooth iff $\forall f \in C^\infty(f), \phi^*(f) \in C^\infty(\mathbb{R}^n)$. The pull back of every smooth function using ϕ is smooth.*

3. $T_x(M) = \{d : C^\infty(M) \to \mathbb{R} \mid d \text{ is linear}, d(f \cdot g) = df \cdot g(X) + f(x) \cdot dg\}$.
   Every d a directional derivation $\Rightarrow$ d is a tangent vector.

4. Define $M_x := \{f \in C^\infty(M) \mid f(x) = 0\}$, and take $T_x(M) := (M_x/M_x^2)^*$.

Exercise 4.2 *Show the definitions are equivalent.*

Now let $\phi : M \to N$ be smooth. The *differential of ϕ in $x \in M$* is $D_x\phi : T_x(M) \to T_{\phi(x)}(N)$ given by the definitions. E.g., $D_x(\phi)(\gamma) := \phi \circ \gamma$ (by (1/2)).

Exercise 4.3 *Show that given manifolds M, N, K and maps $\phi : M \to N, \psi : N \to K, \nu : M \to K$, the differentials have $D_x(\nu) \equiv D_x(\psi) \circ D_x(\phi)$.*

4.2 Types of maps

Now we can define many maps: let $\phi : M \to N$ be a smooth map between manifolds.

- ϕ is an *immersion* if $D_x\phi$ is one-to-one.
- ϕ is a *submersion* if $D_x\phi$ is onto.
- ϕ is a *local isomorphism* or *étale* if $D_x\phi$ is one-to-one and onto.
- ϕ is an *embedding* if it's an immersion and there is a homeomorphism $M \cong \phi(M)$.
- ϕ is a *proper map* if for every compact K, the preimage $\phi^{-1}(K)$ is compact. For example, fibers are compact in M.
- ϕ is a *cover map* if for $x \in N$ there exists a neighbourhood $U \subseteq M$, such that $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is a diffeomorphism, and is a composition $\phi^{-1}(U) \to U \times D \to U$ for a discrete set D.

Example: Let $\phi : [-1, 1] \to \mathbb{R}^2$ be a smooth path that slows to a stop in $\phi(0) = (0, 0)$, but spends no time in $(0, 0)$. That is, all the derivatives are zeroed $\phi^{(n)}(0) = 0$, but $\phi(x) \neq 0$ for all x in some neighbourhood $[-\epsilon, \epsilon]$. Such a ϕ is one-to-one around 0, but is not an immersion at 0.
Example: An immersion isn't necessarily one-to-one. An example is a self-intersecting path $\phi : \mathbb{R} \to \mathbb{R}^2$ with constant speed.
Example: Let L, D be finite dimensional linear spaces. The differential of a map $\phi \in Hom(L, V)$ is ϕ itself. Thus, a one-to-one ϕ will be an immersion, an onto ϕ will be a submersion, and an isomorphism of linear space will be an étale.

Exercise 4.4 *Find a $\phi : M \to N$ which is a one-to-one immersion, but isn't an embedding.*

Exercise* 4.5 *Show that every proper map which a one-to-one immersion is an embedding.*

Exercise* 4.6 *Show that a proper map which is an étale is a cover map, and that a cover map with finite fibers is a proper map and an étale.*

Actually, the above example of a linear map $\phi : L \to D$ is the
Theorem: Let $\phi : M \to N$ be an immersion. Then for every $x \in M$ there is a neighbourhood $U \subseteq M$, and a neighbourhood $V \subseteq N$ of $\phi(x)$, and isomorphisms $\rho : \mathbb{R}^n \to U, \psi : \mathbb{R}^m \to V$, such that the composition $\mathbb{R}^n \xrightarrow{\rho} U \xrightarrow{\phi} V \xrightarrow{\psi^{-1}} \mathbb{R}^m$ is one-to-one.
Alternatively, if ϕ is a submersion then $\rho \circ \phi \circ \psi^{-1}$ is one. If ϕ is an étale the composition will be an homeomorphism

Exercise 4.7 *Prove the theorem using the implicit function theorem.*

Let $\phi : M \to N$ be a submersion. From previous theorem we get that every fiber $\phi^{-1}(x)$ is a submanifold in M. In addition, the tangent space in every point in the fiber will be the kernel of the differential of ϕ. i.e., $\forall y \in \phi^{-1}(x), T_y(\phi^{-1}(x)) = kerD_y\phi$.

When M is a submanifold of $\mathbb{R}^n$, we have the known concept of tangents. Given an immersion $\phi : M \to \mathbb{R}^n$ we have $T_x(M)$ as a subspace of $T_{\phi(x)}(\mathbb{R}^n)$.

Example: Let $M = \mathbb{R}^2, N = \mathbb{R}$, and $\phi(x, y) := x^2 + y^2$. The map ϕ is a submersion away from the origin. The fibers of ϕ are circles/a point/ the empty set – all are submanifolds of M. The tangent space will be as expected in each case.

4.3 Tangent bundles

In the end of the last lesson we defined a *vector bundle* to be a smooth family of vector spaces. For a manifold M we defined a bundle to consists of a vector space for every fiber of a covering map.
We define the tangent bundle of M to be $TM := \{(x, v) \mid x \in M, v \in T_x(M)\}$. As a bundle, it has a projection $p : TM \to M$ defined by $p(x, v) = x$.

Example: Let V be a linear space and let $M \subseteq V$ be an open set in it. By the definition of tangent space for linear spaces, we have $TM \cong M \times V$. So, the tangent bundle will consist of copies of V corresponding to every $x \in M$.
Example: The tangent bundle of $M = S^1$ will be $TS^1 = S^1 \times \mathbb{R}$. In every point the tangent space is one-dimensional, and changes smoothly as we "walk" on the circle. However, on $M = S^2$ the tangent bundle will not be $S^2 \times \mathbb{R}^2$. This holds since every vector field on S^2 vanishes ("you can't comb a hedgehog"), therefore we certainly can't parallelize 2 vector field on it.

4.4 Fun with bundles

A *direct sum* of 2 bundles over M is just the sum of their vector spaces. We can take open neighbourhoods V, W (?) and define $(M \times V) \oplus (M \times W) := M \times (V \oplus W)$. Similarly, we can define tensor products of bundles, multilinear n-forms, and their absolute value and sign.

Exercise 4.8 *Find non-isomorphic bundles E,E', such that $E \oplus F \cong E' \oplus F$ for some bundle F (**Hint:** use vector bundles over S^2).*

Example: We define the Mobius strip by $M := \{(\theta, v) \in S^1 \times \mathbb{R}^2 \mid v \in L_{\frac{\theta}{2}}\}$, where $L_\theta \subset \mathbb{R}^2$ is the line intersecting the x-axis in angle θ. Equivalently, we could define $M_2 := \{(\theta, v) \in S^1 \times \mathbb{R}^2 \mid v \in L_{\frac{\theta}{2}+\frac{\pi}{2}}\}$. The direct sum $M \oplus M_2$ will be $S^1 \times \mathbb{R}^2$. To show that, we can map $(\theta, a) \mapsto (\theta, a_v, a_w)$, where $a = a_v + a_w$ is the decomposition of a.

Let E_1, E_2 be bundles over a manifold M. A map $\phi : E_1 \to E_2$ is defined as a smooth map $M \to M$ that induces a homomorphism of linear spaces in every fiber of the bundle projections.
The kernel and the image of a general map between bundles aren't "nice".

Exercise 4.9 *Show that if $\phi_x : E_1(x) \to E_2(X)$ has constant rank for every $x \in M$, the image and the kernel of ϕ are bundles.*

[??? Missing definition of dual bundle]
Example: The cotangent bundle T^*M is a bundle on M in which $\forall x \in M$, $T^*_x(M) = (T_x(M))^*$.
Example: If a manifold M has the same dimension $dimM := n$ in every point, we can define $\Omega_M^{top} := \Lambda^n(T^*M)$, using the space of n-forms.
Using this, we define the densities and orientations bundles on the manifold, $Dens(M) := |\Omega_M^{top}|, Dens(M) := \text{sign}(\Omega_M^{top})$.

Given a map $\phi : M \to N$ and a bundle $p : E \to M$ we define the *pull-back of E by ϕ* to be $\phi^*(E) := \{(m, v) \in N \times E \mid \phi(m) = p(v)\}$.
Given a submanifold $X \subseteq M$, and an embedding $i : X \to M$, we define the *normal bundle at a point $x \in M$* to be $N_x(M) := i^*(TM)/TX$. Similarly, the *conormal bundle* will be $CN_x(M) := (N_x(M))^*$.
Example: For $M = S^2$ the normal bundle at a point will give the normal vector to it. It will be isomorphic to the trivial bundle on M.

For a submersion $s : M \to N$ we have the *relative tangent bundle* over M, defined as $T_{M/N} := \ker Ds$. The differential $Ds : TM \to s^*(TN)$ is a map between bundles. In every $x \in M$ the relative tangent bundle will be $\ker D_x s$, which is the tangent space to the fiber of $s(x)$.

4.5 Cuts

In set theory, a *cut* of a function $f : X \to Y$ is a function $g : Y \to X$ s.t. $g \circ f \equiv id$. For example, for $f : \mathbb{R}^2 \to \mathbb{R}$ which is the projection $f(x, y) := x$ a cut can be $g(x) := (x, sinx)$. This a just be a (continuous) choice of representatives of fibers.
In our case, cuts of bundles can help us define many basic concepts. For example:

- A cut of the tangent bundle is a vector field.
- A cut of the densities bundle is a differential form.
- A cut of the orientations bundle is an orientation on a manifold.
- A positive cut of the bundle $sym^2(TM)^*$ (=chooses only positive symmetric forms) is a *Riemannian metric* on the manifold.

Exercise 4.10 *Show the every manifold has a Riemannian metric (**Hint:** This cut won't vanish on M. Use partition of unity).*

We won't always have differential forms on a manifold, and the Mobius strip will be a counter-example. However, we can always define densities. This is since density (and metric) are defined using positivity, which is a convex condition. However, the conditions for defining a differential form is a non-vanishing cut, and that's a non-convex condition.

Exercise 4.11 *Let M be a manifold with a Riemannian metric. Define a density over M and describe it explicitly. The density should respect coordinates change, and be the standard density when M is a linear space with the standard metric.*

Since a density over a space gives us a measure on it, we can thus define integrals over manifolds. That is our goal for the next lesson.

5 Lesson 8, 9/12/13

5.1 Recap from last week

- Let X, Y be topological spaces, and $p : E \to X$ a bundle. Last week X, Y were varieties. E is a topological space, and the isomorphism is a homeomorphism of topological spaces. We define a *pullback* of the bundle by $E := \{(x, v) \mid v \in p^{-1}(F(X))\}$, and skip the topology definitions.

  We also demanded that for all $x \in X$ there is a neighborhood $U \subseteq X$, and a diffeomorphism $\phi : U \to U \times \mathbb{R}^n$, and a projection $p_n : U \times \mathbb{R}^n \to U$ such that $p_n \equiv p \circ \phi$.

- A definition of tangent spaces using categories: We'll use a definition from category theory, which only defines the required attributes and doesn't show existence.

  First: A *category* C consists of:

  - A collection[8] of *objects*, denoted $Obj(C)$.
  - For every $X, Y \in Obj(C)$ – a collection of *morphisms*, denoted $Mor(X, Y)$.

  The morphisms in C need to meet some demands, like a composition rule and existence of an identity map.

  A *functor* $F : C \to D$ between two categories consists of:

  - A map between the categories' objects, $Obj(C) \to Obj(D)$.
  - For every $X, Y \in Obj(C)$ – a map between the morphism $Mor(C, D) \to Mor(F(C), F(D))$.

[8] A category can be very big, e.g. the category of all sets. To avoid logical paradoxes like Russell's paradox, we use the terms "collection of objects" or "class of objects".

The map between the morphisms must respect the composition rule and the identity maps of the categories (meaning, $F(Id_A) = Id_{F(A)}$, and $F(\phi) \circ F(\psi) = F(\phi \circ \psi)$). There are numerous examples of categories, such as **Set** (category of all sets) or **Man** (category of all manifolds). We'll denote **Vect** the category of all linear spaces, and **ptMan** the category of "pointed manifolds" (pairs of manifold and a point in them).

The *tangent space* is a functor from the category of pointed manifolds to the category of linear spaces. So, the functor is $T : \mathbf{ptMan} \to \mathbf{Vect}$. For which:

1. The tangent space of a linear space is the space itself, $T_0(V) = V$.
2. If $(V, 0)$,$(W, 0)$ are Euclidean spaces (=normed spaces), and $\phi : (V, 0) \to (W, 0)$ a map between them (continuous? isometry?), then $\lim\limits_{x \to 0} \frac{T(\phi(x)) - \phi(x)}{\|x\|} = 0$.
3. For every $\phi \in Mor(V, W)$ which is an open embedding, the map $T(\phi) \in Mor(T(V), T(W))$ is an isomorphism (This demand might follow from the previous ones).

5.2 Analytic manifolds

We now turn to study manifolds over all local fields, and not only over fields of characteristic 0.
Let (M, O) be a pair of a manifold and a sheaf of functions from M to the field. For every subset $U \subseteq F^n$ and open $V \subseteq U$ there is the local sheaf $O_U(V)$ of functions $\rho : V \to F$. We'll say (M, O) is an *analytic* manifold if every $\rho \in O_U(V)$ is analytic. That means: for every $x \in V$ there is a sequence $\{a_k\}_{k \in \mathbb{N}^{\mathbb{N}}}$ (k is a multi-index), and a neighbourhood $W \subseteq V$ of x, for which $\forall y \in W,\ \rho(y) = \sum\limits_{k \in \mathbb{N}^{\mathbb{N}}} a_k (y - x)^k$.
Remark: Some authors use the term "analytic manifold" for other objects. We use it in the same sense it was used in Serre's book.
Remark: We use the terms "manifold" for a smooth object, a "variety" for a manifold which may have singularities.

Example: *Affine analytic variety* are the zero-set of a given collection of analytic functions $\{f_i\}_{i \in I}$. In order for it to be a manifold, we need to make sure it's smooth. A possible condition is that in any point the differentials $Df_i(x)$ have full rank.
We'll mainly use *affine algebraic manifolds* in the course, which are affine analytic manifolds defined by polynomials $f_i \in \mathbb{C}[X_1, ...X_n]$.

5.3 Comparison of smooth and analytic manifolds

- In a smooth manifold M, every open $U \subseteq M$ "looks like" $\mathbb{R}^n$. That's not the case in analytic manifolds. e.g., there is no smooth map from the unit ball to the space.

- We don't have partition of unity in analytic manifolds. If an analytic function zeroes in some neighbourhood, it must be the zero function.
- In both analytic and smooth cases we can use the implicit function theorem, which will be an important tool in our proofs. In algebraic geometry the conditions for the theorem don't hold.

Over analytic manifolds, we can still define vector bundles and specifically tangent bundles, vector fields and differential forms. However, the absolute value we defined is real, $|\cdot| : F \to \mathbb{R}$. Thus, we cannot define $Dens(V)$ and $Ori(V)$ using $|V| := \{f : V^* \to \mathbb{R} \mid f(kv) = |k| \cdot f(v)\}$. In order to define densities (and later integrals) we'll use sheaves.

5.4 Sheaves

Let X be a topological space. A *sheaf* F over X is a map from the set of all open subsets $U \subseteq X$ to the image $F(U)$. In general, the images can be groups, ring, vector spaces, etc. In our case, $F(U)$ will be spaces of functions, and elements $\xi \in F(U)$, called *section*, will be function $\xi : U \to F$. The sheaf axioms will enable us to glue together coinciding sections $\xi : U \to F, \psi : V \to V$, where $U \cap V \neq \emptyset$. Let's give a formal definition.
Definition: Let X be a topological space. A *sheaf* F consists of:

1. An assignment rule from every open $U \subseteq X$ to a vector space $F(U)$.
2. Restriction rules $res_V^U : F(U) \to F(V)$, with the following properties:
   - basic demands: $F(\emptyset) = \{0\}$, $res_U^U \equiv id$.
   - composition: for open $V \subseteq U \subseteq W$, $res_V^W \equiv res_V^U \circ res_U^W$.

   For every an open cover $U = \bigcup\limits_{i \in I} U_i$:

   - if $\forall i \in I,\ res_{U_i}^U \xi \equiv 0$ for some $\xi \in F(U)$, then $\xi \equiv 0$.
   - if $\forall i, j \in I,\ res_{U_i \cap U_j}^{U_i}(\xi_i) \equiv res_{U_i \cap U_j}^{U_j}(\xi_j)$, then $\exists \xi \in F(U)$ s.t. $\forall i \in I, res_{U_i}^U(\xi) \equiv \xi_i$.

Remark: For $\xi \in F(U)$, We denote $\xi|_{U_i} := res_{U_i}^U(\xi)$.

We already showed sheaves of functions and distributions over spaces. We stated the general definition here to define sheaves from bundles.
Given a bundle $p : E \to X$, we'll define a sheaf $F(X)$. For every open $U \in X$ we'll define $F_E(U) := \{s : U \to p^{-1}(U) \mid p\circ \equiv id\}$, where s is a "suitable"(?) map.
Example: The trivial bundle $E = X \times V$ defines the sheaf $F_E(U) := \{s : U \to V \mid \textit{suitable } s(?)\}$. The resulting sheaf can be called *the free sheaf* on X.

Counter-example: The constant sheaf. Let V be a vector space over a field K. We'd like to define a constant sheaf by $F(U) := V$ for every open $U \subseteq X$. To meet the condition $F(\emptyset) = \{0\}$, we need to define: $F(U) := V$ only for non-empty open subsets. This will create a problem with the restriction map. Given any $W \subseteq U$, the restriction res^U_W will be the identity map on V. The sheaf demands will hold: having $res^U_{U_i} v = 0$ (even for some i) means $id(v) = 0$ and $v = 0$. Also, if $id(v_i) = id(v_j)$ for $v_i \in F(U_i), v_j \in F(U_j)$, we'll take $v = v_i = v_j$ in F(U) for the last axiom. However, for disjoint open $U, W \subseteq X$ we can choose different $u, w \in V$, and get a contradiction: for $u \in F(U), w \in F(W)$ there is no $v \in F(X)$ for which $u = id(v) = id(w)$, as required in the last axiom.
So, the constant map "$F(U) := V$ for every non-empty open $U \subseteq X$" doesn't produce a sheaf. It does produce a *pre-sheaf* – an almost-valid sheaf, except for the last axiom. Every pre-sheaf can be *sheafify* to a unique sheaf. In our case that would result in the sheaf of locally constant functions on X. We'll use that sheaf a lot when working with l-spaces.

Remark: Since every bundle is locally trivial, we can define bundles as "locally trivial sheaves" (?).

5.5 Defining densities on $\mathbb{Q}_p$

Let V be a linear space over a local field F. We can now define $|V|$ over $\mathbb{R}$. Given a one-dimensional bundle $p : E \to X$ we'll define $|E|$ to be the locally constant bundle over $\mathbb{R}$. That is, $|E|(U)$ is the collection of all functions $s_x : x \to p^{-1}(x)$, such that exists a neighbourhood $W \subseteq X$ of x and a trivialization $\phi : p^{-1}(W) \to W \times V$ to the constant bundle.

Exercise 5.1 *The definition doesn't depend on the trivialization ϕ.*
***(Hint:** Choose 2 trivializations ϕ_1, ϕ_2 and show that $\phi_1^{-1} \cdot \phi_2$ is just a product by a scalar. Remember that for a non-zero function f to a local field, the absolute value $|f|$ is locally constant).*

We're interested in the bundle of the upper forms $\Omega^n(X)$, from which we defined the densities sheaf $Dens(X) := |\Omega^n(X)|$. If a form $\omega \in \Omega^n_x(U)$ doesn't zero, we will have $|\omega| \in D_x(U)$. [9]
This way we can integrate the non-zero forms. We'll decompose U into open subsets, on which the bundle Ω^n_x is trivial, and we'll get a constant sheaf $D_x(U)$, whose sections are locally constant functions. Thus, every $x \in X$ has neighbourhood U in which the sections are constant, and $U \cap X$ is diffeomorphic to a linear space (we might need to shrink U farther to make sure we have such diffeomorphism). Using this neighbourhoods we can define our integral as we did with smooth manifolds.

[9] I skipped a definition of continuous section of F, defined $F(U) \bigotimes_{C^\infty(U)} C(U)$.

5.6 Distributional sections of sheaves on l-spaces

Let F be a sheaf over X. For every section $s \in F(X)$ we'll define the support of s to be the complement to the zero set of s, $\text{supp}(s) := X \setminus \bigcup_{s|_U \equiv 0} U$. The sections with a compact support will be denoted $F_c(X)$. For an l-space, the basic sheaf will be the constant one, and the Schwartz functions will be $S(U, F) = F_c(U)$ for that sheaf F. The dual space $S^*(U, F) := S(U, F)^*$ will be the distributions on X.

Now we can define generalized functions over analytic manifolds [10]. Given an analytic manifold X over a local field F, we'll define $C^{-\infty}(U) := S^*(U, D_x)$, when D_x is the bundle of densities. These are the functionals from smooth measures with compact support.
We can introduce an equivalent definition, using smooth measures. There is a Borel measure over every topological space. We want the measure to be also a Radon measure (=finite on compact sets), and to coincide locally with a Haar measure. These demand defines the set of *smooth measures with compact support*. Those are all the measures μ over X, s.t. for every $x \in X$ there is a neighbourhood $W \subseteq X$ for which:

- W is homeomorphic to an open subset in F^n by some homeomorphism ϕ.
- The push-forward of μ coincides (locally) with a Haar measure h over F^n, $\phi_*(\mu|_W) \equiv h|_{F^n}$.

Exercise 5.2 *Show an isomorphism of this measures space and $S(U, D_x)$.*

Next, for any sheaf F over a vector space X we'll define $C^{-\infty}(X, F) := S^*(X, F^* \bigotimes_{C_X^\infty} D_x)$.
This definition involves a tensor product over a ring. We don't want to get into that, so we'll cheat and assume F is locally constant. Since we already showed that D_x is locally compact, there will be a small neighbourhood of every point in which F is a vector space, and we can look at definition as a regular tensor product. The elements in this products are all maps $f : F(U) \to D_x(U)$ such that:

- <u>f is linear over C_X^∞</u>: $\forall \phi \in C_X^\infty(U), s \in F(U) : \ f(\phi \cdot s) = \phi \cdot f(s)$.
- <u>F comes from a locally constant sheaf</u>: for every $x \in U$ there is a neighbourhood $W \subseteq X$ isomorphic to open $W' \subseteq F^n$, and... ?? (we'll continue the definition next week).

5.7 Distributions over smooth manifolds

We defined distributions over a $\mathbb{R}^n$ as the dual space for $C_c^\infty(\mathbb{R}^n)$. However, for a smooth manifold M we'll define directly the generalized functions $C^{-\infty}(M)$.

[10] we won't define generalized functions of l-spaces in this course.

We need to check that given a diffeomorphism $\phi : M \to \mathbb{R}^n$ which is not a linear map, we can still push forward measures and distributions. This way, given any open $U \subseteq M$ isomorphic to $\mathbb{R}^n$ we could define $C^{-\infty}(U) := C^{-\infty}(\mathbb{R}^n)$.
The problem with this kind of definitions is the dependance on ϕ. To show independence of the diffeomorphism, given diffeomorphisms $\phi_1, \phi_2 : U \to \mathbb{R}^n$ we need to show an isomorphism $i_{\phi_1,\phi_2} : C^{-\infty}_{\phi_1}(U) \to C^{-\infty}_{\phi_2}(U)$. In addition, given diffeomorphisms $\phi_1, \phi_2, \phi_3 : U \to \mathbb{R}^n$ we need to show a composition holds $i_{\phi_1,\phi_2} \circ i_{\phi_2,\phi_3} \equiv i_{\phi_1,\phi_3}$. We could then define $C^{-\infty}(U)$ as the set of correspondences $\phi \mapsto C^{-\infty}_{\phi}(U)$ under the equivalence relation generated by the isomorphisms i_{ϕ_1,ϕ_2}.

To switch from generalized functions over open sets in M to generalized functions on the entire manifold we'll use the sheaf structure.
We denote Θ the set of all open sets $U \subseteq M$ that are diffeomorphic to $\mathbb{R}^n$ and define:

$$C^{-\infty}(M) := \{\alpha : \Theta \to C^{-\infty}(U) \mid \forall U \subseteq V \ open : \ res^V_U(\alpha(V)) = \alpha(U)\}$$

Similarly, we'll define generalized functions over a bundle E, which we denote $C^{-\infty}(M, E)$. It can be shown the definition doesn't depend on the trivialization of E. That gives an explicit isomorphism between $C^{-\infty}(X, V)$ and $C^{-\infty}(X, W)$. Specifically, we get a definition for *smooth distributions on a manifold*, $C^{-\infty}(M, D_M)$.

Let's define a topology on $C^{\infty}_C(M)$. We take a locally finite cover $M \subseteq \bigcup_{i\in I} U_i$, and define an onto map $\bigoplus_{i\in I} C^{\infty}_c(U_i) \to C^{\infty}_c(M)$. This could give us a topology on the image from the direct sum. We'll take a different route:
Let ξ be a vector field over M. It is a differential operator on every function $f \in C^{\infty}_c(M)$ (sending $f \mapsto \xi(f)$). Every differential operator $D(f)$ is a sum $\sum_i \xi_i(f) \cdot g_i$, for some functions g_i and vector fields ξ_i. Thus, if f has compact support, the image $D(f)$ will also have compact support. The opposite implication is also correct, and you reader will prove it.

Exercise 5.3 *Let $f \in C^{\infty}(\mathbb{R}^n)$. Show that f has compact support iff $\sup_x |Df(x)| < \infty$ for every differential operator D.*

This enables us to define a topology using a system of semi-norms (like we had in locally convex spaces). Every differential operator D will define the norm $\|d\|_D := \sup_x |Df(x)|$. That gives us a topology on $C^{\infty}_C(M)$.

Exercise* 5.4 *Show that this topology is identical to the one defined by the map $\bigoplus_{i\in I} C^{\infty}_c(U_i) \to C^{\infty}_c(M)$ above.*

Exercise* 5.5 *Show that $C^{-\infty}(M, D_M) \cong C^{\infty}_C(M)^*$.*

Defining differential operators on a field of a bundle:
$T^E_{(X,x)} := \{(D_1, D_2) : C^\infty(X, E) \times C^\infty(X) \to \mathbb{R} \mid D_1(f \cdot s) = D_2(f) \cdot s + f \cdot D_2(s)\}$