Arielle Leitner: Transitions of Geometries and Groups

Most of my research concerns the study of transitions between different *homogeneous spaces,* G/H, associated with a fixed Lie group, G, obtained by taking limits of conjugates of the subgroup H inside the ambient group G. The idea of geometric transition may be studied from the perspectives of geometry, topology, algebraic geometry, and dynamics.

Intuition and Motivation

Imagine blowing up a ball with air so that eventually the ball is so large, it looks like the earth. Locally, the ball looks flat. This example is given in [16]: a sequence of spheres tangent to a plane, with increasing radius, will limit to the tangent plane in the Hausdorff topology on closed sets. Such a process is an example of a *geometric transition*, or a continuous path of geometric structures that changes type in the limit.

There are several ways of making the idea of inflating a ball mathematically precise. Envision the curvature of the ball approaching zero. Or, define a way to measure coordinates on the ball, and then use coordinates to describe the radius of the ball increasing. A sphere is *intrinsically* different from the plane. On a sphere, the angles in a triangle will sum up to more than 180 degrees, since the edges bulge outwards. In the plane, the angles in a triangle sum to exactly 180 degrees. This property about triangles is *intrinsic* to the geometry of the space, and will hold true no matter how large or small the triangle is. Blowing up a ball is an example of a transition between two different kinds of geometry: spherical and Euclidean.

The idea of continuously deforming one kind of geometry into another appears in many areas of mathematics and physics, see [16]. Homogeneous spaces have groups of isometries which are Lie groups, and have an associated Lie algebra, with multiplication given by the bracket $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. This bilinear map is determined by the action on a basis, and hence by *structure constants*. The structure constants may continuously change, as long as $[,]$ still determines a Lie algebra. This is related to the theory of Inönü-Wigner contractions in physics, see [11]. Physicists use deformations of Lie algebras in several ways, for example the "classical limit" in relativity where the speed of light, $c\to\infty$; which recovers Newtonian mechanics. Another example is transitioning from quantum mechanics to Newtonian mechanics, when $\hbar\to 0$.

Thurston conjectured and Perelman proved: every compact 3-dimensional manifold is composed of pieces, each of which has one of 8 kinds of 3-dimensional geometry, two kinds are the familiar spherical and Euclidean geometry, [45]. These *Thurston geometries* are (almost) subgeometries of real projective geometry and one may study geometric transitions in this context as paths of conjugacies, [16]. I study geometric transitions given by *conjugacy limits.*

Definition 1. *Let G be a Lie group. A subgroup, $H\leq G$,* limits under conjugacy *to another subgroup, $L\leq G$, if there is a sequence of conjugating elements, (p_n) in G, such that $p_nHp_n^{-1}\to L$ with respect to the Hausdorff distance.*

Chabauty Space of Limits of the Cartan Subgroup in $SL_3(\mathbb{R})$

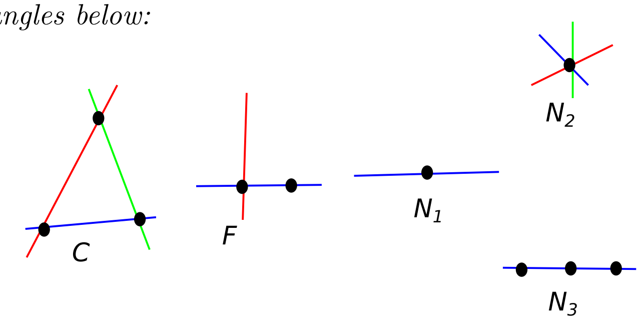
Building on work of Haettel, [26], I studied geometric transitions of the diagonal Cartan subgroup in $\mathrm{SL}_n(\mathbb{R})$ in [36, 38]. For example, when $n=3$, a diagonal matrix with distinct eigenvalues determines a projective triangle, since each eigenvector of the matrix designates a vertex of the triangle. Taking the limit of applying a sequence of projective transformations to a triangle identifies some of the edges or vertices, to form a *degenerate triangle.*

Theorem 2 (Leitner [36]). *1. Any subgroup of* $\mathrm{SL}_3(\mathbb{R})$ *isomorphic to* $\mathbb{R}^2$ *is conjugate to exactly one of the following groups, where* $a, b \in \mathbb{R}_{>0}$ *and* $s, t \in \mathbb{R}$*:*

$$\begin{array}{ccccc} C & F & N_1 & N_2 & N_3 \\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix}, & \begin{pmatrix} a & t & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}, & \begin{pmatrix} 1 & s & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & s & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

2. *Each of these groups is a conjugacy limit of the diagonal Cartan subgroup.*

3. *The set of conjugacy classes of limit groups is in bijection with the set of equivalence classes of degenerate triangles below:*



To prove this theorem, I use the *hyperreals* [23], a non-Archimedean ordered field containing the reals, which contains numbers that are infinitesimally small, and others that are infinitely large. The hyperreals provide a convenient method for describing phenomena that appear after an *infinite* amount of time, by giving a precise way to measure the infinities involved using ratios of hyperreals. This is often more convenient than computing convergence of ratios of sequences.

Future work: I have classified conjugacy limits of the diagonal Cartan subgroup in $\mathrm{SL}_n(\mathbb{R})$ for $n \leq 4$, and it remains open to classify them for $n \geq 5$. There are two possible approaches: studying degenerations of projective n-simplices, or expanding the hyperreal techniques in [36].

Chabauty Space of Limits of the Cartan Subgroup in $\mathrm{SL}_n(\mathbb{Q}_p)$

In joint work with Corina Ciobotaru and Alain Valette [15], we extended some of these results to $\mathrm{SL}_n(\mathbb{Q}_p)$. Instead of acting on projective space, the groups act on the Bruhat-Tits building for $\mathrm{SL}_n(\mathbb{Q}_p)$. We determine conjugacy limits of the diagonal Cartan subgroup up to conjugacy for $n \leq 4$. There are more conjugacy classes of conjugacy limits than in the real case, because $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is larger than $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Our results are as follows: The diagonal group is the center of a stabilizer of an apartment in the building. If a group contains hyperbolic elements, then a flat torus theorem implies it stabilizes a flat in the Bruhat-Tits building. If a group contains no hyperbolic elements, then it is contained in the unipotent radical of a parabolic group that stabilizes a facet in the spherical building at infinity. A necessary condition for a conjugacy limit of groups $H_1 \to H_2$ is that the facet in the spherical building at infinity corresponding to H_2 is contained in the facet for H_1.

Future work: Extend the classification to $n \geq 5$ and generalize to other local fields.

The Topology on the Space of Limit Groups: $\mathit{Red}(n) \subset \mathit{Ab}(n)$

The set of all closed subgroups of a group is a topological space with the *Chabauty topology* on closed sets: [19, 27, 26]. The Chabauty topology is the natural topology for conjugacy limits of groups [16, 14]. Define two topological subspaces with the subspace topology, $\widehat{Ab}(n)$: the set of all subgroups of $\mathrm{SL}_n(\mathbb{R})$ isomorphic to $\mathbb{R}^{n-1}$; and let $\widehat{Red}(n)$ be the space of all limits of the diagonal Cartan subgroup in $\mathrm{SL}_n(\mathbb{R})$. Quotient by conjugacy in $\mathrm{SL}_n(\mathbb{R})$ to obtain two spaces with the quotient topology: $Ab(n)$ and $Red(n)$. Since limit groups of the diagonal Cartan subgroup are isomorphic to $\mathbb{R}^{n-1}$, then $Red(n) \subset Ab(n)$ [31].

Suprenko and Tyshkevitch, [44], classified conjugacy classes of maximal commutative nilpotent subalgebras over $\mathbb{C}$, for $n \leq 6$. They showed $Ab(5)$ is finite, so $Red(5)$ is finite. Iliev and Manivel, [31], ask if $Red(n)$ is finite when $n \geq 6$. I found a conjugacy invariant of groups that shows:

Theorem 3 (Leitner [38])**.** *If $n \geq 7$, then the covering dimension $0 < dim(Red(n)) = O(n^2)$.*

Thus for $n \geq 7$ there are infinitely many non-conjugate limits of the diagonal Cartan subgroup. When $n \leq 5$ there are finitely many conjugacy classes of limits. The case $n = 6$ remains open.

When $n \leq 4$ then $Ab(n) = Red(n)$. Haettel [26] and Iliev and Manivel [31] gave a dimension counting argument which shows $Red(n) \subsetneq Ab(n)$ for $n \geq 7$. In [38] I found the first explicit examples of elements of $Ab(n) - Red(n)$ for $n = 5, 6$, which may be extended for $n \geq 7$.

Theorem 4 (Leitner [38])**.** *$Ab(n) = Red(n)$ if and only if $n \leq 4$. If $n \geq 5$, then $Red(n) \subsetneq Ab(n)$.*

In [15] we extend Theorem 3 to $\mathbb{Q}_p$. The analog of Theorem 4 is computationally more difficult since we cannot apply [44], and need to make new computations for $n = 5, 6$ which is large.

Future research: Over any local field, when is an abelian group a conjugacy limit group? What special properties do conjugacy limit groups have? I hope to find a conjugacy invariant of abelian groups that distinguishes conjugacy limit groups. I plan to explore the the topology of the spaces $Ab(n)$ and $Red(n)$. For example: How many components do $Ab(n)$ and $Red(n)$ have? Does every component of $Ab(n)$ contain a component of $Red(n)$? Is $Red(n)$ a retract of $Ab(n)$?

Varieties of Closed Subgroups

Often it is useful to distinguish conjugacy classes of abelian Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$. A matrix, $A \in \mathrm{GL}_n(\mathbb{C})$, has a Jordan Normal Form (JNF) that uniquely determines the conjugacy class of A. It is possible to simultaneously put *two* matrices in JNF if they commute. Currently, I am studying a Jordan Normal Form Invariant for an *abelian Lie subgroup* G of $\mathrm{GL}_n(\mathbb{R})$. The invariant is a function on the projective space $\mathbb{P}(G)$. Level sets of this function are projective semi-algebraic varieties. The goal is to obtain a practical method to distinguish the conjugacy class of G. Varieties connected to the Jordan Normal Form invariant have long been studied as *rank varieties* and *varieties of commuting matrices*, see [7, 20, 22, 42].

The subspace of conjugates of the diagonal group has closure which is a (semi-algebraic) variety $\mathcal{V}$, called the *Chabauty compactification* of the associated homogeneous space. The Chabauty compactification gives information about the dynamics of the action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathcal{V}$.

For $n = 3$, Haettel showed $\mathcal{V}$ is a CW complex with 2-skeleton the wedge sum of $\mathbb{R}P^2$ and S^2, see [26]. The cells of the CW complex correspond to conjugacy classes of groups, H, with dimension equal to the dimension of the Borel group minus the dimension of the normalizer $N_G(H)$. The attaching maps for cells correspond to limits of groups $H_1 \to H_2$, where the cell corresponding to H_2 has lower dimension than the cell corresponding to H_1.

Future research: Determine Chabauty compactifications in higher dimensions over different fields. In general, one might ask if the Chabauty compactification of G/H and the subalgebra compactification $\mathfrak{g}/\mathfrak{h}$ are homeomorphic. Guivarc'h- Ji-Taylor [25] showed that these are homeomorphic when $H = K$, the maximal compact subgroup. Haettel [26] has shown the Chabauty compactification and Lie algebra compactification are homeomorphic for the diagonal subgroup in $\mathrm{SL}_n(\mathbb{R})$. A natural question is when these compactifications are homeomorphic for different fields and subgroups H. In [15] we give strong evidence that these compactifications are the same for the Cartan subgroup in $\mathrm{SL}_n(\mathbb{Q}_p)$. In a new project with Corina Ciobotaru we are studying limits of H in p-adic groups, where H is the fixed point set of some involution (not the Cartan).

Let G be a Lie group (not necessarily $\mathrm{SL}_n(\mathbb{R})$). Suppose $H \leq G$ is any closed subgroup and L is a conjugacy limit of H. One might ask which properties of a group are preserved under taking a conjugacy limit. For example, if H is reductive (distal or amenable) then is L reductive (distal or amenable)?

Generalized Cusps on Convex Projective Manifolds

Another application of these ideas is to study *generalized cusps* on convex projective manifolds, see [4, 33, 17, 18]. A *convex projective manifold* $C = \Omega/\Gamma$ is the quotient of a convex subset of projective space, Ω, by a discrete group of projective transformations $\Gamma \subset \mathrm{PGL}_{n+1}(\mathbb{R})$. If Ω is in the complement of a hyperplane, then C is a *properly* convex projective manifold. The *holonomy* of a convex projective manifold Ω/Γ is a representation of the fundamental group $\pi_1(\Omega/\Gamma) = \Gamma \to \mathrm{PGL}(\Omega) \subset \mathrm{PGL}_{n+1}(\mathbb{R})$. Properly convex projective manifolds are a generalization of hyperbolic manifolds, and have a rich deformation theory. Contrary to Mostow rigidity for hyperbolic manifolds, [17, 18] show that if the ends of the manifold have the structure of generalized cusps, then it is possible to deform the manifold to make new properly convex projective manifolds.

A *generalized cusp* in dimension 3 is a properly convex projective manifold that is the product of a ray and a torus. The holonomy centralizes a 1-parameter subgroup of $\mathrm{PGL}_n(\mathbb{R})$.

Theorem 5 (Leitner, [37])**.** *A generalized cusp on a properly convex projective 3-dimensional manifold is projectively equivalent to one of 4 possible cusp types.*

Generalized cusps on projective surfaces also give rise to affine structures on the 2-torus, see [24, 39]. For a generalized cusp $C = \Omega/\Gamma$ in dimension n, we require that ∂C is compact and strictly convex (contains no line segment) and that there is a diffeomorphism $h : [0, \infty) \times \partial C \to C$. Together with Sam Ballas and Daryl Cooper in [5] we classified generalized cusps in dimension n, and explored new geometries arising from such cusps.

Theorem 6 (Ballas-Cooper-Leitner [5])**.** *The holonomy of a generalized cusp is a lattice in one of a family of Lie groups $G(\psi)$ parameterized by a point $\psi = (\psi_1, ..., \psi_n) \in \mathbb{R}^n$, with $\psi_1 \geq \cdots \geq \psi_n \geq 0$.*

A generalized cusp may be determined either by its group of isometries, or by a properly convex projective domain. More generally, a maximal-rank cusp in a hyperbolic n-orbifold is determined by the similarity class of lattice in $\mathrm{Isom}(\mathbb{E}^{n-1})$. We parametrize the space of lattices, and use this to describe transitions between cusps. Let $\mathcal{M}od^n$ denote the collection of conjugacy classes of unmarked lattices in holonomy groups of generalized cusps.

Theorem 7 (Ballas-Cooper-Leitner [5])**.** *There is a bijection between elements of $\mathcal{M}od^n$ and projective classes of generalized cusps.*

We show every generalized cusp is foliated by $(n-1)$-dimensional manifolds with a Euclidean structure, and every generalized cusp deformation retracts to a hyperbolic cusp. We also discuss the volume of cusps with respect to the Hausdorff measure induced by the Hilbert metric.

Theorem 8 (Ballas-Cooper-Leitner [5]). *A generalized cusp has finite volume if and only if there exists $k \leq n-2$ such that $\psi_i = 0$ for all $i \geq k$.*

In a second project with Ballas and Cooper [6], we studied *deformations* of generalized cusps. We produce several different invariants for studying the moduli space of marked generalized cusps which is a subspace of the space of conjugacy classes of representations of $\pi_1 C$.

Theorem 9 (Ballas-Cooper- Leitner [6]). *Generalized cusps are determined:*

- *by the* complete invariant*, which is comprised of a character and a positive definite quadratic form β which gives a Euclidean structure on the boundary.*
- *by β together with some weight data subject to a simple geometric constraint.*
- *by the sum of the projective class of β and by a cubic differential on the boundary of C.*

The different descriptions of the moduli space produced by these invariants are homeomorphic.

The complete invariant is a generalization of a character for semisimple representations. The moduli space of generalized cusps is a semi-algebraic set of dimension n^2-n. Every generalized cusp is a geometric limit of diagonalizable cusps. A generalized cusp is finitely covered by a Euclidean manifold. In dimension 3, the final description produces a moduli space of generalized cusps that is homeomorphic to the product of $\mathbb{R}^2$ and a cone on a solid torus. We also determine the topology of the moduli space for higher dimensions.

Future research: Bobb [9] has realized all cusp types, except the diagonalizable type, attached to convex projective manifolds. It remains an open question to realize the diagonalizable cusps attached to a manifold. It is also open to determine which manifolds permit which cusp types, and to study how to deform the manifolds with cusps attached, rather than deforming the cusps alone. Finally, one might consider cusps which are not of maximal rank. Classifying geometric structures on the ends of manifolds is progress towards a higher dimensional Geometrization Theorem.

Chabauty space of subgroups of $\mathrm{SL}_2(\mathbb{R})$

Together with Ian Biringer and Nir Lazarovich [8], we are studying the topology of $\mathrm{Sub}(G)$, the Chabauty space of subgroups of $G = \mathrm{SL}_2(\mathbb{R})$. The connected subgroups are easy to classify and find limits of, building on limits of 1 parameter groups are calculated in [3]. The space of discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ with the Chabauty topology is homeomorphic to the space of vectored hyperbolic orbifolds [14]. To understand limits of discrete subgroups, we parametrize the degeneration of a surface to another surface with lower genus or fewer cusps using *conformal grafting* [10, 28]. Grafting is the process of gluing in a Euclidean cylinder with length tending to infinity that is parametrized in Euclidean coordinates, after which we apply the uniformization map to put a hyperbolic structure on the resulting piecewise hyperbolic/Euclidean surface.

In the first paper with Biringer and Lazarovich [8], we study the spaces of lattices and elementary subgroups of G, and prove a continuity result for conformal grafting of (possibly infinite type) vectored orbifolds that will be useful in both papers. In the future we will write a second instalment where we graft along infinite type vectored orbifolds to understand the remainder of $\mathrm{Sub}(G)$.

First we identify the homotopy type of the space of elementary subgroups of G. Then for a fixed finite type hyperbolizable 2-orbifold S, we show that the space $\mathrm{Sub}_S(G)$ of all lattices $\Gamma < G$ with $\Gamma\backslash\mathbb{H}^2 \cong S$ is a fiber orbibundle over the moduli space $\mathcal{M}(S)$. We describe the closure $\overline{\mathrm{Sub}_S(G)}$ in $\mathrm{Sub}(G)$. The bulk of the paper is taken up by proving the following using grafting:

Theorem 10 (Biringer-Lazarovich-Leitner [8]). *$\partial \operatorname{Sub}_S(G)$ has a neighborhood deformation retract within $\overline{\operatorname{Sub}_S(G)}$.*

The deformation retract enables us to make a van Kampen argument to glue the spaces $\operatorname{Sub}_S(G)$ together, and onto the space of elementary subgroups.

Theorem 11 (Biringer-Lazarovich-Leitner [8]).

1. *When S is not one of finitely many low complexity orbifolds, we show that $\overline{\operatorname{Sub}_S(G)}$ is simply connected.*
2. *When S is a sphere with three total cusps and cone points, we show that $\overline{\operatorname{Sub}_S(G)}$ is a (usually nontrivial) lens space.*

Finally, we show that when $(X_i, v_i) \to (X_\infty, v_\infty)$ is a (possibly infinite type) smoothly converging sequence of vectored hyperbolic 2-orbifolds, and we graft in Euclidean annuli along suitable collections of simple closed curves in the X_i, then after uniformization, the resulting vectored hyperbolic 2-orbifolds converge smoothly to the expected limit. As part of the proof, we give a new lower bound on the hyperbolic distance between points in a grafted orbifold in terms of their original distance.

Future work: Describing the topology of the Chabauty space of a second countable locally compact group is difficult, and the topology is known only for $\mathbb{R}$ and $\mathbb{R}^2$ [29]. There are a few other cases where the topology is reasonably well understood [34, 13]. This project is an advancement towards understanding the topology of Chabauty spaces in general. One might hope to understand the topology of $\operatorname{Sub}(\mathrm{SL}_3(\mathbb{R}))$, and indeed together with Nir Lazarovich we have classified limits of *connected* subgroups of $\mathrm{SL}_3(\mathbb{R})$, [35]. Studying limits of discrete subgroups in higher rank Lie groups is the next step, and is related to the topology of invariant random subgroups [1, 2, 21].

Coarse Groups and Coarse Geometry

Together with Federico Vigolo, I am studying the category of *coarse groups*. Coarse geometry is the study of the large scale geometric features of a space or, more precisely, of those properties that are invariant up to "uniformly bounded error". This language allows us to formalize the idea that two spaces such as $\mathbb{Z}^n$ and $\mathbb{R}^n$ look alike when seen "from very far away". Studying coarse properties of spaces allows us to use topological/analytic techniques on a discrete space like $\mathbb{Z}$ by identifying it with a continuous one like $\mathbb{R}$ or, vice versa, to use discrete/algebraic methods on continuous spaces.

The Milnor-Svarc lemma has a natural reinterpretation to coarse groups. If a group has a bi-invariant metric, then this induces a coarse structure on the group [12]. Coarse groups provide a unifying framework for many ideas from geometric group theory. We are writing a monograph which will lay the category theoretic foundations for coarse groups and open new avenues of research.

A coarse space is a space X with a coarse structure $\mathcal{E}$ which is a collection of subsets of $X \times X$ satisfying certain axioms see [43]. A *coarse group* is a group object in the category of coarse spaces, so that the group laws hold up to "uniformly bounded error." We call a "group" a *set-group* since it is a group object in the category of sets.

In the first part of the monograph we lay a unified ground work for coarse spaces, and develop a theory of coarse groups, coarse subgroups, coarse quotient groups, isomorphism theorems, and coarse actions. We develop our intuition using our example of the free group on two generators with the cancellation metric (unlike the Cayley metric, a generator can be cancelled from anywhere in the word). In the second part of the monograph we explore a variety of topics that we hope will promote future research. Here are some highlights:

- We study when set-groups G admit unbounded coarsely connected coarse structures $(G, \mathcal{E})$.
- We study coarse structures on the integers and show there is a continuum of distinct coarse structures on $\mathbb{Z}$.
- We study cancellation metrics, and put forth a candidate for a coarse group which is not coarsely equivalent to a set-group.
- We study the relationship between commensurators, coarse automorphisms, and outer automorphisms, and give concrete results for $G = F_n$ and $\mathbb{Z}^n$.

Almost-Regular Dessin d'Enfant

A branched covering is a covering map away from finitely many branch points, which determine a branch datum. A branch datum is *compatible* if it satisfies the Riemann-Hurwitz formula restricting the genus, and some orientability conditions. The *Hurwitz existence problem* asks which compatible branch data are realized by branched coverings [30]. A *Dessin d'Enfant* is a connected graph where each vertex is assigned one of two colors and the endpoints of any edge are different colors. Moreover, there is a cyclic ordering of the edges around any vertex. See [41, 46]. The Riemann Existence Theorem says that a branched covering exists if and only if the corresponding Dessin can be drawn. Given a Dessin, we can construct a branch datum by recording the degrees of the vertices and faces.

As a result of Corollary 4.3 in [40], there are 4 finite types of regular ramifications (the ramification degree at each point is equal), these are branched coverings from the torus to the sphere, realized as tilings by hexagons or squares on the torus. We study when it is possible to change the branch data slightly to an *almost-regular ramification type* where the ramification degree over each point is the same except for a bounded number of entries, called the *error*.

Theorem 12 (König-Leitner-Neftin [32])**.** *Almost regular ramification types in genus 0 and 1 with bounded error are realizable, except four types in genus 1, and one nonrealizable type in genus 0.*

Future work: We showed that the genus 1 types are realizable by making changes to only finitely many tiles of a regular tiling of the torus. Answering this question for larger error using the techniques we developed would be a good starting point for an undergraduate research project. It would also be interesting to consider the same question for surfaces of genus ≥ 2.

Answers to the Hurwitz problem are useful in solving problems which reduce to the case of determining the existence of a map between Riemann surfaces with given ramification type, or the existence of branched coverings, or rational functions of certain ramification type.

References Cited

[1] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet, *On the growth of L^2-invariants for sequences of lattices in Lie groups*, Ann. of Math. (2) 185 (2017), no. 3, 711-790

[2] Miklos Abert, Yair Glasner, Balint Virag, and et al, *Kesten's theorem for invariant random subgroups*, Duke Mathematical Journal 163 (2014), no. 3, 465-488.

[3] Baik, H., and Clavier, L. *The Space of Geometric Limits of One-generator Closed Subgroups of* $\mathrm{PSL}_2(\mathbb{R})$ Algebr. Geom. Topol., Vol 13, Issue 1 (2013), pp. 549-576.

[4] Ballas, S. *Finite Volume Properly Convex Deformations of the Figure Eight Knot*. Geom. Dedicata 178 (2015) 49–73

[5] Ballas, S., Cooper, D., Leitner, A. *Generalized Cusps in Convex Projective Manifolds: Classification* arXiv:1710.03132 To appear in J. Topology

[6] Ballas, S., Cooper, D., Leitner, A. *The Moduli Space of Marked Generalized Cusps in Real Projective Manifolds* https://arxiv.org/pdf/2008.09553.pdf

[7] Basili, R. *On the Irreducibility of Varieties of Commuting Matrices*, J. Pure Appl. Algebra 149 n. 2 (2000) 107-120.

[8] Biringer, Ian, Lazarovich, Nir, and Leitner, Arielle. *On the Chabauty space of* $PSL(2\mathbb{R})$*, I: lattices and grafting*, arXiv preprint arXiv:2110.14401, (2021)

[9] Bobb, M. D. *Convex projective manifolds with a cusp of any non-diagonalizable type.* J. London Math. Soc. (2019) doi:10.1112/jlms.12208

[10] Bourque, M. *Conformal Grafting and Convergence of Fenchel-Nielsen Twist Coordinates*, Conformal Geometry and Dynamics of the Amer. Math. Soc. 19 (2015), no. 1, 1-18.

[11] Burde, D. *Contractions of Lie algebras and algebraic groups.* Arch. Math. (Brno), 43(5):321-332, 2007.

[12] M. Brandenbursky, S. Gal, J. Kedra and M. Marcinkowski, *Cancelation norm and the geometry of bi-invariant word metrics,* Glasgow Mathematical Journal, 58 (2016), no. 1, 153-176.

[13] M. Bridson, P. de la Harpe, and V. Kleptsyn, *The Chabauty Space of Closed Subgroups of the Three Dimensional Heisenberg Group*, Pacific J. Math 240 (2009), 1-48.

[14] R. Canary, D. Epstein, and P. Green, *Notes on Notes of Thurston, Analytical and Geometrical Aspects of Hyperbolic Spaces*, 1987.

[15] Ciobotaru, C., Leitner, A., and Valette, A., *Chabauty Limits of Subgroups of* $SL_n(\mathbb{Q}_p)$. Submitted https://arxiv.org/pdf/1711.04864.pdf

[16] Cooper, D., Danciger, J., and Wienhard, A. *Limits of Geometries.* Trans. of Amer. Math Soc. 370 (2018), 6585-6627.

[17] Cooper, D., Long, D., and Tillmann, S. *On Convex Projective Manifolds and Cusps*, Advances in Mathematics 277 (2015), 181-251.

[18] Cooper, D., Long, D., and Tillman, S., *Deforming Convex Projective Manifolds* Geom. Topol. 22, 3, (2018), 1349-1404.

[19] Chabauty, C. *Limite d'Ensembles et Géométrie des Nombres.* Bull. Soc. Math. France, 78:143-151, 1950.

[20] Eisenbud, D., and Saltman, D., "Rank Varieties of Matrices" in *Commutative Algebra.* Berkley, CA, 1987. 173-212. Mathematical Sciences Research Institute Publications, New York: Springer, 1989.

[21] Tsachik Gelander, *A view on Invariant Random Subgroups and Lattices*, arXiv preprint arXiv:1807.06979 (2018).

[22] Gerstenhaber, M. *On Dominance and Varieties of Commuting Matrices,* Ann. Math. 73 (1961) 324-348.

[23] Goldblatt, R. *Lectures on the Hyperreals: An Introduction to Non-standard Analysis*. New York: Springer-Verlag, 1991.

[24] Goze, M., and Remm, E. *Affine Structures on Abelian Lie Groups* Linear Algebra and its Applications 360 (2003) 215-230, 2002.

[25] Guivac'h, Y., Ji, L., and Taylor, J., *Compactifications of Symmetric Spaces* Progress in Mathematics Volume 156, 1998

[26] Haettel, T. *Compactification de Chabauty de l'Espace des Sous-groupes de Cartan de* $\mathrm{SL}_n(\mathbb{R})$*.* Mathematische Zeitschrift, Springer-Verlag, 2012.

[27] de la Harpe, P. *Spaces of Closed Subgroups of Locally Compact Groups.* Arxiv 0807.2030v2 submitted 2008.

[28] Hensel, S. *Iterated Grafting and Holonomy Lifts of Teichmüller Space*, Geom. Ded. 155 (2011), no.1, 31-67.

[29] P. Hubbard and I. Pourezza, *The Space of Closed Subgroups of* $\mathbb{R}^2$, Topology 18 (1979), 143-146.

[30] Hurwitz, A., *Über Riemannsche Flächen mit gegebenen Verzweigungspunkten.* Math. Ann. 39 (1891), 1-60.

[31] Iliev, A.; Manivel, L. *Varieties of Reductions for* gl_n. Projective Varieties with Unexpected Properties, 287-316, Walter de Gruyter, Berlin, 2005.

[32] König, J., Leitner, A., and Neftin, D. *Dessin d'Enfant on a Torus* Topology and Applications, 243 (2018) 78-99.

[33] Koszul, J., *Déformations de connexions localement plates.* Ann. Inst. Fourier (Grenoble), 18 (1968):103-114.

[34] B. Kloeckner, *The Space of Closed Subgroups of* $\mathbb{R}^n$ *is Stratified and Simply Connected*, J. Topology (2009), 57–588.

[35] Lazarovich, N., Leitner, A., *Local Limits of Connected Subgroups of* $\mathrm{SL}(3,\mathbb{R})$ submitted, https://arxiv.org/pdf/1911.09491.pdf

[36] Leitner, A. *Conjugacy Limits of the Cartan Subgroup in* $\mathrm{SL}_3(\mathbb{R})$*.* Geom. Ded. (2016) 180: 135.

[37] Leitner, A. *A Classification of Generalized Cusps on Properly Convex Projective 3-Manifolds.* Topology and its Applications 206, (2016), 241-254

[38] Leitner, A. *A Continuum of Non-Conjugate Limits of the Cartan Subgroup in* $\mathrm{SL}_n(\mathbb{R})$*.* Proc. Amer. Math. Soc. 144 (2016) 3243-3254.

[39] Nagano, T., and Yagi, K. *Affine Structures on the Real Two-Torus* Bul. Amer. Math. Soc., Vol 79, No. 6, 1973.

[40] Neftin, D., and Zieve, M. *Monodromy Groups of Indecomposable Coverings* in preparation

[41] Pervova, E., and Petronio, C. *On the existence of branched coverings between surfaces with prescribed branch data, I* Algebraic & Geometric Topology 6 (2006) 1957-1985

[42] Richardson, R. *Commuting Varieties of Semisimple Lie Algebras and Algebraic Groups*, Compositio Math. 38 (1979) 311-327.

[43] Roe, J. *Lectures on Coarse Geometry*, vol. 31, American Mathematical Soc., 2003.

[44] Suprenko, D., Tyshkevitch, R. *Commutative Matrices.* Academic Press, New York, NY, 1968.

[45] Thurston, W. *Three-dimensional Geometry and Topology*, Princeton Univ. Press, Princeton, NJ, 1997.

[46] Zapponi, L. *What is...a dessin d'enfant?* Notices Amer. Math. Soc. 50(7) (2003) 788-789.