1. Reflected BM

(a) For a regular BM we know that $p(y, t|x, 0) = p(B_t = y|B_0 = x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$. For the reflected BM we may write

$$p(y, t|x, 0) = p(B_t = y|B_0 = x) + p(B_t = -y|B_0 = x)$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} + \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y+x)^2}{2t}}$$

$$= \frac{1}{\sqrt{2\pi t}} \left( e^{-\frac{(y-x)^2}{2t}} + e^{-\frac{(y+x)^2}{2t}} \right)$$

$$= \frac{2}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \cosh \left( \frac{xy}{t} \right)$$

For $y = 0$ we have that $p(y = 0, t|x, 0) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \neq 0$, however for $y < 0$ the probability $p(y, t|x, 0) = 0$ (it is a reflected motion), thus we have discontinuity at $y = 0$.

(b) The heat equation is of the form

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}$$

Deriving the reflected BM

$$\frac{\partial p}{\partial t} = \frac{-1}{\sqrt{8\pi t^3}} \left( e^{-\frac{(y-x)^2}{2t}} + e^{-\frac{(y+x)^2}{2t}} \right) + \frac{1}{\sqrt{2\pi t}} \left( \frac{(y-x)^2}{2t^2} e^{-\frac{(y-x)^2}{2t}} + \frac{(y+x)^2}{2t^2} e^{-\frac{(y+x)^2}{2t}} \right)$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \left( -\frac{1}{t} \cosh \left( \frac{xy}{t} \right) + \frac{y^2 + x^2}{t^2} \cosh \left( \frac{xy}{t} \right) - \frac{xy}{t^2} \sinh \left( \frac{xy}{t} \right) \right)$$

The spatial first derivative

$$\frac{\partial p}{\partial y} = \frac{2}{\sqrt{2\pi t}} \left( -\frac{y}{t} \right) e^{-\frac{y^2}{2t}} \cosh \left( \frac{xy}{t} \right) + \frac{2}{\sqrt{2\pi t}} \frac{e^{-\frac{y^2}{2t}}}{t^2} \sinh \left( \frac{xy}{t} \right)$$

It is easy to see that for $y = 0$ this derivative vanishes ($\sinh(0) = 0$). The second derivative

$$\frac{\partial^2 p}{\partial y^2} = \frac{2}{\sqrt{2\pi t}} \frac{y^2 - t}{t^2} e^{-\frac{y^2}{2t}} \cosh \left( \frac{xy}{t} \right) + \frac{2}{\sqrt{2\pi t}} \left( -\frac{y}{t} \right) e^{-\frac{y^2}{2t}} \left( \frac{x}{t} \right) \sinh \left( \frac{xy}{t} \right) + \frac{2}{\sqrt{2\pi t}} \left( \frac{x}{t} \right) \left( -\frac{y}{t} \right) e^{-\frac{y^2}{2t}} \sinh \left( \frac{xy}{t} \right) + \frac{2}{\sqrt{2\pi t}} \frac{e^{-\frac{y^2}{2t}}}{t^2} \left( \frac{x^2}{t^2} \right) \cosh \left( \frac{xy}{t} \right)$$

$$= \frac{2}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \left( \frac{y^2 - t + x^2}{t^2} \cosh \left( \frac{xy}{t} \right) - \frac{xy}{t^2} \sinh \left( \frac{xy}{t} \right) \right)$$
It is easy to see that the reflected BM solves the heat equation \( \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial y^2} \) with the boundary condition \( \frac{\partial u}{\partial y}|_{y=0} = 0 \).

2. Recurrence in BM

Consider \( \Pr (B_t \geq a | \tau_a \leq t) \) that is the probability of the BM to be above \( a \) at time \( t \) provided the first passage time \( \tau_a \leq t \). Since \( \tau_a \leq t \) there exist a time \( s \leq t \) s.t. \( B_s = a \) therefore the two events are equivalent \( \Pr (B_t \geq a | \tau_a \leq t) = \Pr (B_t \geq a | B_s = a) \). From the properties of BM we know that \( \Pr (B_t \geq a | B_s = a) = \frac{1}{2} \).

We may write

\[
\frac{1}{2} = \Pr (B_t \geq a | B_s = a) = \Pr (B_t \geq a | \tau_a \leq t) = \frac{\Pr (B_t \geq a, \tau_a \leq t)}{\Pr (\tau_a \leq t)} = \frac{\Pr (\tau_a \leq t | B_t \geq a) \cdot \Pr (B_t \geq a)}{\Pr (\tau_a \leq t)}
\]

Since BM is continuous the probability \( \Pr (\tau_a \leq t | B_t \geq a) = 1 \). We know that \( \Pr (B_t \geq a) = \frac{1}{\sqrt{2\pi t}} \int_a^{\infty} e^{-\frac{y^2}{2t}} \, dy \).

Putting everything together we obtain

\[
\frac{1}{2} = \frac{\Pr (B_t \geq a)}{\Pr (\tau_a \leq t)} = \frac{\Pr (\tau_a \leq t | B_t \geq a) \cdot \Pr (B_t \geq a)}{\Pr (\tau_a \leq t)} = \frac{1 \cdot \Pr (B_t \geq a)}{\Pr (\tau_a \leq t)}
\]

\[
\Pr (\tau_a \leq t) = \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} e^{-\frac{y^2}{2t}} \, dy
\]

Note: For \( s = t \) the probability \( \Pr (B_t \geq a | B_s = a) \neq \frac{1}{2} \), however this event is of measure zero thus it does not affect the result for \( \Pr (\tau_a \leq t) \).

Looking at the limit

\[
\lim_{t \to \infty} \Pr (\tau_a \leq t) = \lim_{t \to \infty} \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} e^{-\frac{y^2}{2t}} \, dy
\]

It may be viewed as the area under the graph of the normal distribution curve as the variance increase. Figure 1 depicts in black the required area, the dotted region is the area of

\[
\lim_{n \to \infty} \frac{2}{\sqrt{2\pi t}} \int_0^{a} e^{-\frac{y^2}{2t}} \, dy
\]

The height of the rectangle (dotted plus gray areas) equals to \( a \cdot \frac{2}{\sqrt{2\pi t}} \). For any variance \( t \) we know that the value of the integral \( \frac{2}{\sqrt{2\pi t}} \int_0^{\infty} e^{-\frac{y^2}{2t}} \, dy = 1 \). As \( t \to \infty \) the dotted area bounded by \( a \cdot \frac{2}{\sqrt{2\pi t}} \) tends to zero, while the total area (dotted plus black) is constant, therefore we may conclude that

\[
\lim_{t \to \infty} \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} e^{-\frac{y^2}{2t}} \, dy = 1
\]

Thus the probability of BM arriving at \( a \) for some time \( t \) starting from \( B_0 = 0 \) equals to one.
Figure 1: Evaluating \( \lim_{t \to \infty} \frac{1}{\sqrt{2\pi t}} \int_{a}^{\infty} e^{-\frac{y^2}{2t}} \, dy \)

Considering the mean \( \mathbb{E}\{\tau_a\} \) we may compute it through its density function \( p(\tau_a = t) = \frac{d}{dt}\Pr(\tau_a \leq t) \) and the expectancy is

\[
\mathbb{E}\{\tau_a\} = \int_{0}^{\infty} t \cdot p(\tau_a) \, dt = \int_{0}^{\infty} t \cdot \frac{d}{dt}\Pr(\tau_a \leq t) \, dt
\]

Changing variables and computing the derivative (thank you Leibniz)

\[
\frac{d}{dt}\Pr(\tau_a \leq t) = \frac{d}{dt} \left\{ \frac{2}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\frac{y^2}{2t}} \, dy \right\} = \frac{2}{\sqrt{2\pi}} \frac{d}{dt} \left\{ e^{-\frac{y^2}{2t}} \right\} = \frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2t}} \frac{d}{dt} \left\{ \frac{a}{\sqrt{t}} \right\} = \frac{3}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}
\]

Computing the mean

\[
\mathbb{E}\{\tau_a\} = \int_{0}^{\infty} t \cdot \frac{d}{dt}\Pr(\tau_a \leq t) \, dt = \int_{0}^{\infty} \frac{3t}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}} \, dt
\]

\[
= \int_{0}^{\infty} \frac{3}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \, dt
\]

The term \( e^{-\frac{a^2}{2t}} \) tends to 1 as \( t \to \infty \) thus there exist \( s \) s.t. for \( t > s \); \( e^{-\frac{a^2}{2t}} > \frac{1}{3} \) and therefore \( \frac{3}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} > \frac{1}{\sqrt{2\pi t}} \).

We know that \( \int_{0}^{\infty} \frac{1}{\sqrt{2\pi t}} \, dt \to \infty \) for any \( \alpha > 0 \). Hence

\[
\mathbb{E}\{\tau_a\} = \int_{0}^{\infty} \frac{3}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \, dt \geq \int_{0}^{\infty} \frac{1}{\sqrt{2\pi t}} \, dt \to \infty
\]

Therefore the mean of \( \tau_a \) is infinite.

One dimensional BM experience self similarity, therefore if \( \lim_{t \to \infty} \Pr(\tau_a \leq t) = 1 \) then the probability of reaching from any \( B_{t_1} = a_1 \) to \( B_{t_2} = a_2 \) tends to 1 for large enough \( t \).

\[
\Pr(B_t(\omega) = x \text{ for some } t > T) = \int_{x_T} \Pr(B_t(\omega) = x | B_T = x_T) \cdot \Pr(B_T = x_T) \, dx_T
\]

\[
= \int_{x_T} 1 \cdot \Pr(B_T = x_T) \, dx_T = 1
\]

Hence one dimensional BM is recurrent.
3. Absorbed BM

Let the pdf of absorbed BM at \( a \) be denoted by \( p_a (y, t|x, 0) \). The pdf of regular BM is \( p(y, t|x, 0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \) for the absorbed BM we need to exclude from this term paths that reach the point \( y \) at time \( t \) but cross \( a \) on the way. These paths have the pdf of \( \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-2a)^2}{2t}} \), i.e. they are as likely as paths that starts from \( y_o = 2a \) at \( t_o = 0 \) and reach \( y \) at time \( t \) (I reflect the segment of the path between 0 and \( a \) around the line \( y = a \)). Thus the pdf of absorbed BM is

\[
p_a (y, t|x, 0) = \begin{cases} \\
\frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} - \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-2a)^2}{2t}} & \text{for } y(t) < a \\
2 \frac{1}{\sqrt{2\pi t}} a e^{-\frac{y^2}{2t}} \ dy' & \text{for } y(t) = a \\
0 & \text{for } y(t) > a 
\end{cases}
\]

That is for \( y(t) < a \) we exclude the paths that cross \( y = a \) for some time \( s < t \). The probability of \( y(t) = a \) is exactly \( \Pr (\tau_a \leq t) \), i.e. we reached \( a \) at some time \( s \leq t \) and now the BM is absorbed at \( a \).

4. Stochastic integrals

Looking at two possible definitions of Stochastic integrals

\[
S_1 [f] = \lim_{n \to \infty} \sum \int f \left( B \left( \frac{t_j + t_{j+1}}{2}, t_j \right) \right) \cdot \left[ B_{t_{j+1}} - B_{t_j} \right]
\]

\[
S_2 [f] = \lim_{n \to \infty} \sum \int f \left( \frac{B_{t_{j+1}} + B_{t_j}}{2}, t_j \right) \cdot \left[ B_{t_{j+1}} - B_{t_j} \right]
\]

Since \( B_t \) is continuous w.p. 1 the limits \( \lim_{\Delta t \to 0} B \left( \frac{t_j + t_{j+1}}{2}, t_j \right) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} (B_{t_{j+1}} + B_{t_j}) \) as depicted in figure 2. As \( n \to \infty \) the values \( \lim_{n \to \infty} \int f \left( \frac{B_{t_{j+1}} + B_{t_j}}{2}, t_j \right) \) (continuity of \( f(x,t) \) and \( B(t) \)). Therefore \( S_1 [f] = S_2 [f] \) w.p. 1. Let \( t_{j+\frac{1}{2}} \) denote the point \( t_{j+\frac{1}{2}} = \frac{t_j + t_{j+1}}{2} \). The behavior of the differences is characterized by BM properties

\[
B \left( t_{j+\frac{1}{2}} \right) - B \left( t_j \right) \sim N \left( 0, \frac{1}{2} \Delta t \right)
\]

\[
B \left( t_{j+1} \right) - B \left( t_{j+\frac{1}{2}} \right) \sim N \left( 0, \frac{1}{2} \Delta t \right)
\]

And their difference

\[
B \left( t_{j+1} \right) - 2B \left( t_{j+\frac{1}{2}} \right) + B \left( t_j \right) \sim N \left( 0, \Delta t \right)
\]

Looking at Taylor expansion of the function \( f(x,t) \approx f \left( x_o, t_o \right) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\Delta x)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x \partial t} (\Delta x) \Delta t + \frac{1}{8} \frac{\partial^3 f}{\partial t^2} (\Delta t)^2 \). Taking the expansion for each interval separately (note that the term \( \Delta t = 0 \))

\[
f \left( B \left( t_{j+\frac{1}{2}} \right), t_j \right) = f \left( B \left( t_j \right), t_j \right) + \frac{\partial f}{\partial x} \bigg|_{B(t_j),t_j} \cdot \left( B \left( t_{j+\frac{1}{2}} \right) - B \left( t_j \right) \right) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \bigg|_{B(t_{j+\frac{1}{2}},t_j)} \cdot \left( B \left( t_{j+\frac{1}{2}} \right) - B \left( t_j \right) \right)^2
\]

\[
f \left( \frac{B_{t_{j+1}} + B_{t_j}}{2}, t_j \right) = f \left( B \left( t_j \right), t_j \right) + \frac{1}{2} \frac{\partial f}{\partial x} \bigg|_{B(t_j),t_j} \cdot \left( B_{t_{j+1}} - B_{t_j} \right) + \frac{1}{8} \frac{\partial^2 f}{\partial x^2} \bigg|_{B(t_{j+1},t_j)} \cdot \left( B_{t_{j+1}} - B_{t_j} \right)^2
\]
The terms \( \left( B \left( t_{j+\frac{1}{2}} \right) - B (t_j) \right)^2 \) and \( (B_{t_{j+1}} - B_{t_j})^2 \) may be neglected. They represent a normal RV squared which has variance of \( o \left( (\Delta t)^2 \right) \), as \( \Delta t \to 0 \) this term decays faster to 0. Looking at the difference:

\[
S_1 [f] - S_2 [f] = \lim_{n \to \infty} \sum_j \left( f \left( B \left( t_{j+\frac{1}{2}} \right), t_j \right) - f \left( \frac{B_{t_{j+1}} + B_{t_j}}{2}, t_j \right) \right) \cdot [B_{t_{j+1}} - B_{t_j}]
\]

\[
= \lim_{n \to \infty} \sum_j \left( \frac{1}{2} \left. \frac{\partial f}{\partial x} \right|_{B(t_j), t_j} \left( B \left( t_{j+1} \right) - 2B \left( t_{j+\frac{1}{2}} \right) + B \left( t_j \right) \right) \right) \cdot [B_{t_{j+1}} - B_{t_j}]
\]

It is easy to see that \( B \left( t_{j+1} \right) - 2B \left( t_{j+\frac{1}{2}} \right) + B \left( t_{j} \right) \sim N \left( 0, \Delta t \right) \) thus it behaves like \( o \left( \sqrt{\Delta t} \right) \). The term \( \Delta B_j \sim N \left( 0, \Delta t \right) \), and behaves like \( o \left( \sqrt{\Delta t} \right) \) as well, thus their product behaves like \( o \left( (\Delta t)^{\frac{3}{2}} \right) \). As \( n \to \infty \) the time step \( \Delta t \to 0 \) and the product tends to zero. Hence the two terms approximate the same integral: \( S_1 [f] = S_2 [f] \).

Taking the 2\(^{nd}\) argument of \( f(x, t) \) at different part of the interval \( \Delta t_j \) will not change the equality \( S_1 [f] = S_2 [f] \). Since it is fixed along the interval \( \Delta t_j \) the \( \Delta t \) argument for the Taylor expansion will remain 0. Moreover the value of \( S [f] \) will not be changed due to different choice of the 2\(^{nd}\) argument. Looking at the two sums (for some smooth function \( g(x, t) \))

\[
I_1 [g] = \lim_{n \to \infty} \sum_j g \left( B_{t_j}, t_j \right) \cdot [B_{t_{j+1}} - B_{t_j}]
\]

\[
I_2 [g] = \lim_{n \to \infty} \sum_j g \left( B_{t_j}, t_{j+1} \right) \cdot [B_{t_{j+1}} - B_{t_j}]
\]
Using Taylor expansion

\[ T_2 [g] = \lim_{n \to \infty} \sum_{j} g(B_{t_j}, t_{j+1}) \cdot [B_{t_{j+1}} - B_{t_j}] \]

\[ = \lim_{n \to \infty} \sum_{j} \left[ g(B_{t_j}, t_j) + \frac{\partial g}{\partial t}_{B_{t_j}, t_j} \Delta t + \frac{\partial^2 g}{\partial t^2}_{B_{t_j}, t_{j+t}} \Delta t^2 \right] \cdot \Delta B_j \]

As \( n \to \infty \) the time step \( \Delta t \to 0 \) thus we may neglect the last two terms in the summand and obtain

\[ T_2 [g] = \lim_{n \to \infty} \sum_{j} g(B_{t_j}, t_j) \cdot [B_{t_{j+1}} - B_{t_j}] = T_1 [g] \]

Therefore the exact choice of the point at which the second argument of \( f(x, t) \) is computed does not affect the result of the integral.