Upper bounds for dimension of the global attractor for dissipative dynamical systems with applications to turbulence models

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Introduction

➢ Our motivation – estimation on the number of degrees of freedom in infinite dimensional dissipative systems.

➢ Different concepts of dimension – definitions and examples.

➢ Upper bound for dimension of attractor – statement of the result by P. Constantin and C. Foias.

➢ Examples – Navier-Stokes equation in 2D, Sabra shell model of turbulence.

➢ Sketching the proof of the formula.
Turbulence is the manifestation of the spatio-temporal chaotic behavior of fluid flows at large Reynolds numbers, i.e. of a strongly nonlinear dissipative system with extremely large number of degrees of freedom.

Reynolds number is defined as \( R = \frac{UL}{\nu} \), were \( U, L \) are characteristic velocity and scale and \( \nu \) is a kinematic viscosity.

Kolmogorov theory asserts, that high Reynolds number 3D turbulence generates structures, “eddies”, down to the size of the cut-off length \( L_d \). Below this scale viscosity effectively consumes all the energy flowing from larger scales. A complete specification of the state of a turbulent velocity field must then resolve excitations down to this scales (at least). Hence, in a 3D cube of size \( L \) we have

\[
N = \left( \frac{L}{L_d} \right)^3 \sim R^{9/4}
\]

numbers of degrees of freedom.
Figure 1: Two-dimensional image of turbulent motion at $R = 4000$
Poincarè provides a good discussion of what constitutes a “dimension,” developing the intuitive idea of dimension inductively in terms of “cuts.” Any segment of a curve is isolated (bounded) when it is cut by two points, thus a line segment has dimension one. An area can be cut into isolated areas by a curve, and thus is assigned dimension two. And so on.

Brower (1913) and Lebesgue (1912) gave first topologically invariant definition of dimension.

Let $X$ be a topological space. Then the Lebesgue covering dimension of $X$, or topological dimension of $X$, denoted by $D_T(X)$, equals to the least integer $n$ such that every finite open cover of $X$ has a refinement by a finite open cover of $X$ of order $\leq n + 1$; in case there is no such an integer $n$, we define $D_T(X) = \infty$.

Let $C$ be the open covering of $X$ and $x \in X$. Define $O_C(x)$ - the number of elements of $C$ containing $x$. Then $\max_{x \in X} \{O_C(x)\}$ is the order of the covering $C$. 

Topological dimension
Hausdorff measure

Consider $X$ – a subset of some metric space $H$. For $d \in \mathbb{R}_+$ and $\epsilon > 0$ define

$$\mathcal{H}^d_\epsilon (X) = \inf \sum_i r_i^d,$$

where the infimum is taken over all coverings of $X$ by a subsets of diameter at most $\epsilon$.

$\mathcal{H}^d_\epsilon$ is a non-decreasing function of $\epsilon$, hence the limit as $\epsilon$ goes to 0 exists. We define the $d$-dimensional Hausdorff measure of $X$ as

$$\mathcal{H}^d (X) = \lim_{\epsilon \to 0} \mathcal{H}^d_\epsilon (X)$$
Properties of the Hausdorff measure

Hausdorff measure satisfies the following:

- Defined for all subsets of the metric space and coincides with the Lebesgue measure for Lebesgue measurable sets.

- Scale invariant. For every $\lambda \in \mathbb{R}_+$:

  $$\mathcal{H}^d(\lambda X) = \lambda^d \mathcal{H}^d(X).$$

- Invariant under isometries.
Hausdorff dimension and fractals

Consider $X$ – a subset of some metric space $H$. There exists a unique number $d_0$, such that

$$
\mathcal{H}^d(X) = \begin{cases}
\infty, & d < d_0, \\
0, & d > d_0.
\end{cases}
$$

The number $d_0$ is called the Hausdorff dimension of $X$, denoted by $D_H(X)$. It satisfies

$$
D_T(X) \leq D_H(X).
$$

$X$ is a fractal set if its Hausdorff dimension exceeds its topological dimension (B. Mandelbrot, 1975) namely $D_H(X)$.

"Fractal is a neologism coined by the author (B. Mandelbrot) in 1975 from the Latin adjective fractus, which is related to the verb frangere, “to break”. This coinage responds to the need for a term to denote a mathematical set or a concrete object whose form is extremely irregular and/or fragmented at all scales.”
Figure 2: Koch curve. Hausdorff dimension $= \frac{\ln 4}{\ln 3}$.
Fractal or box-counting dimension

This dimension is accounted in literature under different names. To name a few: box-counting dimension, entropy dimension, capacity dimension, information dimension, etc.

Consider $X$ – a a subset of some metric space $H$. Denote $N_\epsilon(X)$ – the smallest number of balls of radius at most $\epsilon$ needed to cover $X$. The fractal dimension of $X$ is

$$D_F(X) = \limsup_{\epsilon \to 0} \frac{-\log N_\epsilon(X)}{\log \epsilon}.$$ 

Sometimes, the term dimension is referred to the case, where the limit exists and the quantity just defined is called the upper box-counting dimension.
Fractal vs. Hausdorff dimensions

In the case of Hausdorff dimension, for every $\epsilon$ we cover the set $X$ by the balls of radius $\leq \epsilon$. In the definition of the fractal dimension we may cover $X$ with the balls of precisely radius $\epsilon$. Hence,

$$D_H(X) \leq D_F(X).$$

As an example of a strict inequality consider the set $Y = \left\{ \frac{1}{p} : p \in \mathbb{N} \right\}$. In that case

$$0 = D_H(Y) < D_F(Y) = \frac{1}{2}.$$
Let us consider the equation

\[ \frac{du}{dt}(t) = F(u(t)), \quad u(0) = u_0 \]

in some Hilbert space \( H \) and assume that it is well-posed, namely the solution \( u(t) \) exists and is unique for every \( u_0 \in H \) and \( t \geq 0 \).

For every \( u_0 \in H \), \( S(t)u_0 \) is the solution of the equation. The mapping \( S(t) : H \to H \), which satisfies the semigroup property \( (S(t + s) = S(t) \cdot S(s) \) for every \( t, s \geq 0 \), is continuous.

Consider

\[ \frac{dU}{dt}(t) = F'(u(t)) \cdot U, \quad U(0) = \xi \]

the linearized equation around \( u(t) \).
An attractor of the semigroup $\{S(t)\}_{t \geq 0}$ is a set $\mathbb{A} \subset H$ with the following properties:

☞ $\mathbb{A}$ is an invariant set: $S(t)\mathbb{A} \subseteq \mathbb{A}$.

☞ $\mathbb{A}$ possesses an open neighborhood $U \subseteq H$ such that for every $u_0 \in U$, $S(t)u_0$ converges to $\mathbb{A}$ as $t \to \infty$:

$$\text{dist}(S(t)u_0, \mathbb{A}) \to 0, \quad t \to \infty.$$  

$\mathbb{A}$ is a global or universal attractor for $\{S(t)\}_{t \geq 0}$ if it is a compact attractor which attracts all the bounded sets of $H$. 

B. Levant, Upper bounds for dimension of the global attractor for dissipative dynamical systems with applications to turbulence models.
Fix $m \in \mathbb{N}$ and let $U_1, \ldots, U_m$ be the solution of linearized equation corresponding to the initial conditions $\xi_1, \ldots, \xi_m$. Let $Q_m = Q_m(t)$ be the projection of $H$ onto the space spanned by $U_1, \ldots, U_m$.

Denote

$$q_m(t) = \sup_{u \in X} \sup_{|\xi_j| \leq 1} \frac{1}{t} \int_0^t \text{Trace}(F'(u(\tau) \circ Q_m(\tau))d\tau,$$

$$q_m = \limsup_{t \to \infty} q_m(t).$$

**Theorem 1.** Let $\mathbb{A}$ be the global attractor of the semigroup $\{S(t)\}_{t \geq 0}$. If for some $m$

$$q_m < 0,$$

then

$$D_H(\mathbb{A}) \leq D_F(\mathbb{A}) \leq 2m.$$
Navier-Stokes equation

We will consider NSE in the 2-D cube $\Omega = [0, L_0]^2$ with the periodic boundary conditions:

$$\begin{cases} 
\frac{du}{dt} - \nu \triangle u + (u \cdot \nabla u) = f - \nabla p, \\
\nabla \cdot u = 0.
\end{cases}$$

Projecting it on the space of divergence free vector fields we obtain the following abstract form of the equation

$$\frac{du}{dt} + \nu Au + B(u, u) = f,$$

where $A = P(-\triangle)$ is a self-adjoint positive linear operator with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $B(u, u) = P(u \cdot \nabla u)$ is a nonlinear operator. We are working in the Hilbert space $H$ with the norm $|\cdot| = (\cdot, \cdot)$. We also define the space $V = D(A^{1/2})$ with the corresponding norm $\|\cdot\| = ((\cdot, \cdot))$ and $\|u\| = |A^{1/2}u|$. 

Linearization of the Navier-Stokes equation around some function $u \in H$ yields

$$\frac{dU}{dt} + \nu AU + B(u, U) + B(U, u) = 0,$$

and in the notations of the previous slides

$$F'(u) = -(\nu A(\cdot) + B(u, \cdot) + B(\cdot, u)).$$
In order to bound the fractal dimension of the attractor of the 2-D NSE we need to estimate $Trace(F'(u) \circ Q_m)$, where $Q_m$ denotes the orthogonal projector in $H$ onto some $m$ dimensional subspace. Let $\phi_j, j = 1, \ldots, m$ denote the orthogonal basis of that subspace, then substituting the value of $F'(u)$ we get

$$Trace(F'(u) \circ Q_m) = \sum_{i=1}^{m} (F'(u)\phi_i, \phi_i) = -\nu \sum_{i=1}^{m} \|\phi_i\|^2 - \sum_{i=1}^{m} (B(\phi_i, u), \phi_i).$$

The first summand can be bounded from below by use of the estimate $\lambda_j \sim c\lambda_1 j^{2/n}$ as $j \to \infty$, ($n = 2$ - the dimension of the space) and the second is bounded by the inequalities for the non-linear term. Finally we obtain:

$$Trace(F'(u) \circ Q_m) \leq -\nu c_3 \lambda_1 m^2 + \frac{c_2}{2\nu} \|u\|^2.$$
Define the energy dissipation flux $\epsilon$, which could be interpreted as 

$$
\epsilon = \nu < |\nabla u| >^2 = \nu \lambda_1 \limsup_{t \to \infty} \frac{1}{t} \int_0^t \|u(\tau)\|^2 d\tau.
$$

According to the Kolmogorov theory of turbulence, the cut-off length $L_d$ (Kolmogorov dissipation scale) depends only on the viscosity and the energy flux. By dimensional consideration we derive 

$$
L_d^4 = \nu^3 \epsilon^{-1}.
$$

Hence, the dimension of the global attractor of the NSE in 2-D is $2m$, while 

$$
m \geq c_5 \left( \frac{L_0}{L_d} \right)^2 = c_5 \left( \frac{\epsilon}{\nu^3} \right)^{1/2} \lambda_1^{-1}.
$$
Global attractor for Navier-Stokes equations

Using the trace bound and the definition of the energy flux we obtain by dividing by $t$ and taking the limit when $t \rightarrow \infty$

$$q_m = \limsup_{t \rightarrow \infty} \sup_{u \in A} \sup_{|\phi_j| \leq 1} \frac{1}{t} \int_0^t \text{Trace}(F'(u) \circ Q_m)(\tau)d\tau \leq -\nu c_3 \lambda_1 m^2 + c_4 \frac{\epsilon}{\nu^2 \lambda_1}.$$ 

We need to find an $m$ such that the last quantity is less than 0.

Hence, the dimension of the global attractor of the NSE in 2-D is $2m$, while

$$m \leq c_5 \left( \frac{L_0}{L_d} \right)^2 = c_5 \left( \frac{\epsilon}{\nu^3} \right)^{1/2} \lambda_1^{-1}.$$
Once again consider NSE, this time in the 3-D cube $\Omega = [0, L_0]^2$ with the periodic boundary conditions:

$$\begin{cases}
\frac{du}{dt} - \nu \Delta u + (u \cdot \nabla u) = f - \nabla p, \\
\nabla \cdot u = 0.
\end{cases}$$

Spacial Fourier transform of the velocity field (and similarly of the pressure and the forcing)

$$u_k(t) = \int_{\Omega} u(x, t) e^{-ik \cdot x} d^3x.$$ 

NSE in Fourier space takes the form

$$\frac{d}{dt} u_k(t) = -\nu k^2 u_k(t) + \frac{i}{L_0^3} \sum_{k' + k'' = k} (u_{k'}(t) \cdot k'' u_{k''}(t)) - ikp_k(t) + f_k(t),$$

where $p_k$ and $f_k$ are corresponding Fourier modes of the pressure and external force.
The shell models of turbulence are useful phenomenological models that mimics certain features of the Navier-Stokes equation (NSE). The idea behind it is to replace the fluctuation of a turbulent field in an octave of wave numbers $\lambda^n < |k_n| \leq \lambda^{n+1}$, for some parameter $\lambda > 1$, by one or a few representative variables. The range of wave numbers is called a shell, and the variables are called shell variables. Inspired by the NSE, the time evolution of the shell velocities is governed by an infinite system of coupled ordinary differential equations with quadratic nonlinearities, with forcing applied to the large scales and viscous dissipation effecting the smaller ones. Therefore, the shell models could be viewed as a drastic modification of the original NSE in the Fourier space.

Shell models have much less number of degrees of freedom than the original Navier-Stokes equation and hence are very useful in numerical studies of such phenomena as an energy cascade mechanism, statistical properties of the turbulent flow, etc.
The “Sabra” shell model of turbulence

This model was introduced by V. S. L’vov, E. Podivilov, A. Pomyalov, I. Procaccia and D. Vandembroucq from Weizmann institute. The equations are

\[
\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}^* + bk_nu_{n+1}^* - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \quad (1)
\]

for \( n = 1, 2, 3, \ldots \), were the coefficients \( a, b \) and \( c \) are real and the boundary conditions are \( u_{-1} = u_0 = 0 \). The conservation of energy in the inviscid and unforced case \( (\nu = f_n = 0, \text{ for all } n) \) is obtained if \( a + b + c = 0 \). The frequencies \( k_j \) are taken to be

\[
k_j = k_0 \lambda^j, \quad (2)
\]

for some \( k_0 > 0, \lambda > 1 \) and \( \nu > 0 \). In an analogy to the Navier-Stokes equations \( \nu \) represents a kinematic viscosity and \( f \) is a forcing term. Moreover, although the equation does not capture any geometry, we will consider \( L = k_0^{-1} \) as a fixed typical length scale.
Abstract formulation

We write the system in the following functional form

\[
\frac{du}{dt} + \nu Au + B(u, u) = f \tag{3a}
\]

\[
u(0) = u^\text{in}(x), \tag{3b}
\]

in a Hilbert space \(H = \ell_2\). For every \(u, v \in H\), the scalar product \((\cdot, \cdot)\) and the corresponding norm \(|\cdot|\) are defined as

\[
(u, v) = \sum_{n=1}^{\infty} u_n v^*_n, \quad |u| = \left(\sum_{n=1}^{\infty} |u_n|^2\right)^{1/2}.
\]

We denote by \(\{\phi_j\}_{j=1}^{\infty}\) the standard canonical orthonormal basis of \(H\), i.e. all the entries of \(\phi_j\) are zero except at the place \(j\) it is equal to 1.
The linear operator $A : D(A) \to H$ is a positive definite, diagonal operator defined through its action on the elements of the canonical basis of $H$ by

$$A\phi_j = k_j^2 \phi_j.$$ 

We denote $V = D(A^{1/2})$ a Hilbert space equipped with a scalar product

$$((u, v)) = (A^{1/2}u, A^{1/2}v) = \sum_{n=1}^{\infty} k_n^2 u_n v_n^*, \quad \forall u, v \in D(A^{1/2}).$$

The nonlinear operator $B(u, v)$ will be defined formally in the consistent way. Let $u, v \in H$ be of the form $u = \sum_n u_n \phi_n$ and $v = \sum_n v_n \phi_n$. Then

$$B(u, v) = -i \sum_n \left( a k_{n+1} v_{n+2}^* u_{n+1}^* + b k_n v_{n+1} u_{n-1}^* + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} v_{n-1} u_{n-2} \right) \phi_n.$$
Sabra model - linearized equation

The linearized equation takes the form

\[
\frac{dU}{dt} + \nu AU + B_0(u(t))U = 0 \tag{4a}
\]

\[
U(0) = U^{in}, \tag{4b}
\]

where \( B_0(u(t))U = B(u(t), U(t)) + B(U(t), u(t)) \) is a linear operator. In order to simplify the notation, we will denote

\[
F'(u(t)) = -\nu A - B_0(u(t)).
\]
Energy flux and Kolmogorov dissipation length

In analogy with Kolmogorov’s mean rate of dissipation of energy in turbulent flow we define

$$\epsilon = \nu \left\langle \| u \| ^2 \right\rangle ,$$

the mean rate of dissipation of energy in the Shell model system. $\left\langle \cdot \right\rangle$ represents the ensemble average, which in our case should be interpreted as

$$\epsilon = \nu \limsup_{t \to \infty} \frac{1}{t} \int_0^t \| u(s) \| ^2 ds.$$

We will also define the viscous dissipation length scale $L_d$. According to Kolmogorov’s theory, it should only depend on the viscosity $\nu$ and the mean rate of energy dissipation $\epsilon$. Hence, pure dimensional argument lead to the definition

$$L_d = \left( \frac{\nu^3}{\epsilon} \right)^{1/4}.$$
Trace estimate

Let $U_j(t)$ be solutions of the above system satisfying $U_j(0) = U_j^{in}$, for $j = 1, 2, \ldots, N$. Assume now, that $U_1^{in}, U_2^{in}, \ldots, U_N^{in}$ are linearly independent in $H$, and consider $Q_N(t)$ – an $H$-orthogonal projection onto the span $\{U_j(t)\}_{j=1}^N$. Let $\{φ_j(t)\}_{j=1}^N$ be the orthonormal basis of the span of $\{U_j(t)\}_{j=1}^N$.

Using the definition of $F'(u(t))$ and the inequalities for the non-linear term we get

$$Re(Trace[F'(u(t)) \circ Q_N(t)]) = Re \sum_{j=1}^{N} (F'(u(t))φ_j, φ_j) =$$

$$= \sum_{j=1}^{N} -ν\|φ_j\|^2 - Re(B(φ_j, u(t)), φ_j) \leq$$

$$\leq -νk_0^2λ^2 \lambda^{2N} - 1 \frac{λ^{2N} - 1}{λ^2 - 1} + CN\|u(t)\|,$$

for some constant $C > 0$ depending on parameters of the model.
Using the above estimates we finally obtain the bound for the dimension of the global attractor $\mathbb{A}$ of the Sabra shell model of turbulence:

$$D_H(\mathbb{A}) \leq D_F(\mathbb{A}) \leq 4 \log_\lambda \left( \frac{L}{l_d} \right) + 2 \log_\lambda \left( C \frac{\lambda^2 - 1}{\lambda^2} \right) - 2.$$  

This bound agrees well with the physical intuition.
Let $H$ be an arbitrary Hilbert space, $X \subset H$ a compact subset and a continuous mapping $S : H \to H$ satisfying

$$SX = X.$$ 

Assume, that $S$ is uniformly differentiable on $X$. Namely, for each $u \in X$ there exists a linear continuous operator $L(u) \in \mathcal{L}(H, H)$, such that

$$\sup_{0 < |u - v| < \epsilon} \frac{|Su - Sv - L(u)(u - v)|}{|u - v|} \to \epsilon \to 0 0.$$ 

Assume also, that the operator $L(u)$ is uniformly bounded for every $u \in X$

$$\sup_{u \in X} \|L(u)\|_{\mathcal{L}(H,H)} < \infty.$$
If $L$ is a linear compact operator on $H$, $B(0, 1)$ is a unit ball in $H$, then $LB(0, 1)$ is an ellipsoid with semi-axes $\alpha_j(L), j = 1, 2, \ldots.$

The numbers $\alpha_1(L) \geq \alpha_2(L) \geq \ldots$ are eigenvalues of the nonnegative, compact, self-adjoint operator $(L^*L)^{1/2}$.

Let $d = n + s$, $n \in \mathbb{N}$ and $0 \leq s < 1$. Then we define

$$\omega_d(L) = \alpha_1(L) \ldots \alpha_n(L) \cdot \alpha_{n+1}(L)^s = \omega_{n+1}^s \omega_n^{1-s}.$$  

In the case where $L$ is not compact we will define $\alpha_j(L)$ to be

$$\alpha_j(L) = \sup_{\dim F = j} \inf_{\varphi \in F} \frac{|L\varphi|}{|\varphi|},$$

and all the results will be the same.
Hausdorff dimension of an invariant set

Denote

\[ \bar{\omega}_l = \sup_{u \in X} \omega_l(L(u)). \]

**Theorem 2.** Suppose, that for some \( d > 0 \), the operator \( L(u) \) satisfies

\[ \bar{\omega}_d < 1, \]

then the Hausdorff dimension of \( X \) is finite and satisfies

\[ D_{\mathcal{H}}(X) \leq d. \]
Theorem 3. Suppose, that for some $0 < d = n + s$, where $n \in \mathbb{N}$ and $0 \leq s < 1$, the operator $L(u)$ satisfies

$$(\tilde{\omega}_{n+1})^{\frac{d-l}{n+1}}\tilde{\omega}_l < 1, \quad \text{for } l = 1, 2, \ldots, n.$$ 

then the fractal dimension of $X$ is finite and satisfies

$$D_F(X) \leq d.$$
Distortion of the volume

Consider $S$ – a continuous operator acting on a Hilbert space $H$. Consider $u_0 \in H$ and let $L(u_0)$ be the differential of $S$ at the point $u_0$, namely for every $\xi \in H$ of norm 1 and $\epsilon > 0$

$$S(u_0 + \epsilon \xi) = Su_0 + \epsilon L(u_0) \cdot \xi + o(\epsilon).$$

Hence,

$$\frac{|S(u_0 + \epsilon \xi) - Su_0|}{\epsilon} \leq \|L(u_0)\|_{\mathcal{L}(H)} = \alpha_1(L(u_0)), \quad \epsilon \to 0.$$
Similarly, let $\xi_1, \ldots, \xi_m$ be $m$ unit vectors in $H$ and let $\epsilon > 0$. The distortion of the volume of the $m$-parallelepiped spanned by the vectors $u_0, u_0 + \epsilon \xi_1, \ldots, u_0 + \epsilon \xi_m$ under the action of $S$, as $\epsilon \to 0$ tends to

$$\frac{|L(u_0)\xi_1 \wedge \cdots \wedge L(u_0)\xi_m|}{|\xi_1 \wedge \cdots \wedge \xi_m|}.$$ 

It could be shown, that the last quantity is always bounded by $\omega_m(L(u_0))$.

The numbers $\omega_m(L(u_0))$ indicate the largest distortion of an infinitesimal $m$-dimensional volume produced by $S$ around the point $u_0$. 
The proof goes in three stages, while the main one is the iterated covering:

Fix $\varepsilon > 0$ and cover $X$ with the finite number of balls $B(u_i, r_i)$ of radius $r_i < \varepsilon$. By the invariance of $X$ we have

$$X \subseteq \bigcup_i S(B(u_i, r_i) \cap X).$$

Because of the integrability of $S$ we have

$$|Su_i - Sv - L(u_i)(v - u_i)| \leq \eta|v - u_i|,$$

hence, each body $SB(u_i, r_i)$ is included in the sum

$$Su_i + E_i + B(0, \eta).$$

By slightly increasing the axes of $E_i$, we may assume that $E_i + B(0, \eta) \subseteq E_i'$, where the new ellipsoid $E_i'$ is also small – $\omega_d(E_i') < 1$. 
Counting the number of balls of radius \( r_i < \epsilon/2 \) needed to cover each of the ellipsoids \( E'_i \) we may prove that

\[
\mathcal{H}^d_{\epsilon/2}(X) \leq \frac{1}{2} \mathcal{H}^d_\epsilon(X).
\]

It follows that

\[
\mathcal{H}^d(X) = 0,
\]

and it exactly means

\[
D_{\mathcal{H}}(X) \leq d.
\]
Lorenz introduced his model in 1965 by considering the Fourier transform of the Boussinesq convection problem and setting all the modes except the first three to 0. The system is

\[
\begin{align*}
\dot{x} + \sigma x - \sigma y &= 0 \\
\dot{y} + \sigma x + y + xz &= 0 \\
\dot{z} + bz - xy &= -b(r + \sigma).
\end{align*}
\]

Let \( u = (x, y, z) \in \mathbb{R}^3 \), then we can show that the solution \( u(t) \) remains bounded as \( t \to \infty \):

\[
\limsup_{t \to \infty} |u(t)| \leq \frac{b(r + \sigma)}{2\sqrt{b - 1}} = \rho.
\]
Example – the Lorenz model (linearization)

The differential of the mapping $S(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $u_0 \mapsto u(t)$, at $u = (x, y, z)$ is the linear mapping $L(t, u)$, whose value on every $\xi \in \mathbb{R}^3$ is the solution $U(t)$ of the linearized system

$$U' + A(u)U = 0,$$

$$U(0) = \xi,$$

where

$$A(u) = AU + B(u)U$$

$$A = \begin{pmatrix} \sigma & -\sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & b \end{pmatrix}, \quad B(u) = \begin{pmatrix} 0 & 0 & 0 \\ z & 0 & x \\ -y & -x & 0 \end{pmatrix}.\]
Example – the Lorenz model (trace formula)

If $U_1, U_2, U_3$ are three solutions of the linearized equation corresponding to $U_j(0) = \xi_j \in \mathbb{R}^3$ and $Q$ - projector on the space spanned by $U_1(t), U_2(t)$, then by direct calculations we derive

$$\frac{d}{dt}|U_1 \wedge U_2 \wedge U_3|^2 + 2|U_1 \wedge U_2 \wedge U_3|^2 \cdot \text{Trace}(A(u)) = 0.$$  

$$\frac{d}{dt}|U_1 \wedge U_2|^2 + 2|U_1 \wedge U_2|^2 \cdot \text{Trace}(A(u) \circ Q) = 0.$$  

Hence,

$$\omega_3(L(t, u)) = \sup_{|\xi_j| \leq 1} |U_1 \wedge U_2 \wedge U_3| \leq e^{-(\sigma + b + 1)t},$$

and

$$\omega_2(L(t, u)) = \sup_{|\xi_j| \leq 1} |U_1 \wedge U_2| \leq e^{-\int_0^t \text{Trace}(A(u) \circ Q)} \leq e^{-(k_2 - \delta)t},$$

where $k_2 = -(\sigma + b + 1) + m + \frac{1}{2}\rho$.  

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Recall, that by definition if \( d = 2 + s, 0 \leq s < 1 \), then \( \omega_d = \omega_3^s \omega_2^{1-s} \). In order to apply Theorem 1 we need to find a \( d \) which satisfies \( \bar{\omega}_d < 1 \).

Using previous bounds we obtain \( \omega_d(L(t, u)) < 1 \) for every \( u \), whenever \( d = 2 + s \) and \( s \) satisfies

\[
-s(\sigma + b + 1) + (1 - s)k_2 < 0.
\]

Hence,

\[
d \leq 2 + \frac{k_2}{\sigma + b + 1 + k_2}.
\]

For the values of parameters, that were used by E. N. Lorentz in his original work \( \sigma = 10, r = 28 \) and \( b = 8/3 \) we conclude that the Hausdorff dimension of the global attractor \( A \) is bounded by

\[
D_H(A) \leq 2.538.
\]
Lyapunov numbers and Lyapunov exponents

- The Lyapunov numbers of the mapping $S(t)$ at the point $u_0$ are defined as

$$\lambda_j(u_0) = \lim_{t \to \infty} \left( \alpha_j(L(t, u_0)) \right)^{1/t}, \quad j \in \mathbb{N},$$

when the limit exists. When considering $u_0 \in X$, where $X$ is an invariant set of $S$, then those numbers exist $d\nu$ almost everywhere, while $d\nu$ is the invariant measure of $S$ supported on $X$.

- The corresponding Lyapunov exponents are

$$\mu_j(u_0) = \log \lambda_j.$$
Global (uniform) Lyapunov numbers and exponents

For every $t > 0$ and $j \in \mathbb{N}$ define

$$\bar{\omega}_j(t) = \sup_{u \in X} \omega_j(L(t, u)).$$

It can be shown, that $\bar{\omega}_j(t)$ is a subexponential function, namely $\bar{\omega}_j(t + s) \leq \bar{\omega}_j(t) \cdot \bar{\omega}_j(s)$.

Hence, by the properties of subexponential functions, the following limit exists

$$\lim_{t \to \infty} (\bar{\omega}_j(t))^{1/t} = \inf_{t > 0} (\bar{\omega}_j(t))^{1/t},$$

and we denote it $\Pi_j$.

$\Rightarrow$ The uniform Lyapunov numbers of the mapping $S u_0$ are defined recursively as

$$\Lambda_1 = \Pi_1, \quad \text{and for } j \geq 2, \quad \Lambda_j = \Pi_j / \Pi_{j-1}.$$

$\Rightarrow$ The corresponding uniform Lyapunov exponents are

$$\mu_j = \log \Lambda_j.$$
**Theorem 4.** Let $X \subset H$ be the invariant set of $S$. Assume, as before, that $S$ is uniformly differentiable on $X$ with the differential $L(t,u)$ that is uniformly bounded on $X$. Let $\mu_1 \geq \mu_2 \geq \ldots$ be global Lyapunov exponents of $S$.

If for some $n \geq 0$,

$$\mu_1 + \cdots + \mu_n < 0,$$

then

$$\mu_{n+1} < 0, \quad \frac{\mu_1 + \cdots + \mu_n}{|\mu_{n+1}|} < 1,$$

and the Hausdorff dimension of $X$ satisfies

$$D_H(X) \leq n + \frac{\mu_1 + \cdots + \mu_n}{|\mu_{n+1}|}.$$

The fractal dimension of $X$ satisfies

$$D_F(X) \leq (n + 1) \max_{1 \leq l \leq n} \left(1 + \frac{|\mu_1 + \cdots + \mu_l|}{|\mu_1 + \cdots + \mu_{n+1}|}\right).$$
Consider the abstract equation in some Hilbert space $H$

$$\frac{du}{dt} = F(u(t)), \quad u(0) = u_0.$$ 

$S(t)$ - the one parameter family of operators on $H$, which maps $u_0 \in H$ into solution $u(t) = S(t)u_0$ of the equation. Consider the linearized (around $u(t)$) equation

$$\frac{dU}{dt} = F'(u(t))U, \quad U(0) = \xi,$$

and the differential $L(t, u)$, which maps $\xi$ into solution of the linearized equation.
Application of the theorem - II

Let $U_1, \ldots, U_m$ be solutions of the linearized equation with initial conditions $\xi_1, \ldots, \xi_m$. Then

$$\frac{1}{2} \frac{d}{dt} |U_1 \wedge \cdots \wedge U_m|^2 = |U_1 \wedge \cdots \wedge U_m|^2 \cdot \text{Trace}(F'(u(t) \circ Q_m)).$$

Hence, the volume element of the our original evolution equation evolves like

$$|U_1 \wedge \cdots \wedge U_m|^2 = |\xi_1 \wedge \cdots \wedge \xi_m|^2 \exp \left( \int_0^t \text{Trace}(F'(u(\tau) \circ Q_m)) d\tau \right).$$

Using the last equation we can estimate

$$\omega_m(L(t, u)) = \sup_{|\xi_j| \leq 1} |U_1 \wedge \cdots \wedge U_m| \leq \sup_{|\xi_j| \leq 1} \exp \left( \int_0^t \text{Trace}(F'(u(\tau) \circ Q_m)) d\tau \right).$$
Application of the theorem - II

Let us define

\[ q_m(t) = \sup_{u \in X} \sup_{|\xi_j| \leq 1} \frac{1}{t} \int_0^t \text{Trace}(F'(u(\tau) \circ Q_m)) d\tau. \]

\[ q_m = \lim_{t \to \infty} sup q_m(t). \]

Then, by definition of the global Lyapunov exponents

\[ \frac{1}{t} \log \bar{\omega}_m(t) \leq q_m(t) \implies \mu_1 + \cdots + \mu_n \leq q_m. \]

Hence, according to the Theorem, if for some \( m \geq 0 \)

\[ q_m < 0 \implies D_H \leq D_F \leq 2m. \]
Thank you!
Subexponential (subadditive) functions argument

The following lemma is a useful technical argument used, for example, in showing that the uniform Lyapunov numbers exist

Lemma 5. Let $\varphi$ be a subexponential function from $\mathbb{R}_+$ to $\mathbb{R}_+$, namely,

$$\varphi(t + s) \leq \varphi(t)\varphi(s), \quad \forall t, s \in \mathbb{R}_+,$$

such that for some $0 < a < b < \infty$

$$\sup_{t \in [a, b]} \varphi(t) < \infty.$$

The the limit of $\left(\varphi(t)\right)^{1/t}$ as $t$ goes to infinity exists and satisfies

$$\lim_{t \to \infty} \left(\varphi(t)\right)^{1/t} = \inf_{t > 0} \left(\varphi(t)\right)^{1/t}.$$