Energy and Enstrophy Cascades in the Fully Developed Two-Dimensional Turbulence

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Navier-Stokes equations

It is widely believed that all the information about turbulence is contained in the Navier-Stokes equations for a viscous, incompressible, homogenous flow

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \triangle u + (u \cdot \nabla u) = f - \nabla p, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

We will consider the above equations in dimension 2, equipped with periodic boundary conditions in $\Omega = [0, L]^2$. The velocity $u(x,t)$ is a periodic function in each direction with period L. It satisfies for every t

$$\int_\Omega u(x,t)dx = 0.$$

Functional form of the NSE

It is common to consider the Navier-Stokes equations in the form:

$$\frac{du}{dt} + \nu A u + B(u,u) = f, \quad u \in H, \tag{1}$$

where the phase space H is defined as

$$H := \{u \in L^2(\Omega)^2 \ : \ \nabla \cdot u = 0, \quad \int_\Omega u(x)dx = 0\}.$$

The scalar product in H is taken to be

$$(u,v) = \int_\Omega u(x) \cdot v(x) dx,$$

and the norm

$$|u| = (u,u)^{1/2} = \left(\int_\Omega u(x) \cdot u(x) dx \right)^{1/2}.$$

The Stokes operator A

Denote by $\mathcal{P}$ – the Helmholtz-Leray projection from $L^2(\Omega)^2$ onto H.

The linear operator $A = \mathcal{P}(-\triangle)$ is positive definite and self-adjoint. It's eigenvalues are of the form

$$\left(\frac{2\pi}{L}\right)^2 k \cdot k, \quad k \in \mathbb{Z}^2 \setminus \{0\}.$$

We denote by $0 < \lambda_0 = (\frac{2\pi}{L})^2 \leq \lambda_1 \leq \lambda_2 \leq \ldots$. These eigenvalues arranged in an increasing order and counted according to their multiplicities.

Let $w_0, w_1, w_2, \ldots$ denote the corresponding normalized ($|w_j| = 1$, for $j = 0, 1, 2, \ldots$) eigenvectors.

The bilinear operator B

The bilinear operator B is defined as

$$B(u, v) = \mathcal{P}((u \cdot \nabla)v),$$

where $\mathcal{P}$ is the Helmholtz-Leray projection from $L^2(\Omega)^2$ onto H.

The powers of the Stokes operator A

For every $\alpha \in \mathbb{R}$ we define the powers A^α of A by its action on the eigenvectors $\{w_j\}_{j=0}^{\infty}$

$$A^\alpha w_j = \lambda_j^\alpha w_j, \quad \text{for} \quad j = 0, 1, 2, \ldots .$$

For every $\alpha \in \mathbb{R}$ the operator A^α is bounded from the space $D(A^\alpha)$ to H, where

$$D(A^\alpha) = \{u \in H \ : \ \sum_{j=0}^{\infty} \lambda_j^{2\alpha}(u, w_j)^2 < \infty\}.$$

We set $V = D(A^{1/2})$ and the norm on V is

$$\|u\| = |A^{1/2}u| = \left(\int_\Omega \sum_{j=1}^{2} \frac{\partial}{\partial x_j} u(x) \cdot \frac{\partial}{\partial x_j} u(x) \right)^{1/2} = \left(\sum_{j=0}^{\infty} \lambda_j (u, w_j)^2 \right)^{1/2}.$$

Fourier transform in H

Given the periodic boundary conditions, we may express an element in H as a Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{i k_0 k \cdot x},$$

where

$$k_0 = \lambda_0^{1/2} = \frac{2\pi}{L}.$$

Moreover, it follows from the definition of H that $\hat{u}_0 = 0$, and $\hat{u}_k^* = \hat{u}_{-k}$. Moreover, the incompressibility condition implies $k \cdot \hat{u}_k = 0$.

We associate to each Fourier mode $\hat{u}_k$ the corresponding wavenumber $k_0|k|$. The characteristic length scale of $\hat{u}_k$ will be denoted by $l_k = \frac{1}{k_0|k|}$.

Projectors

For $k \geq 0$ we define the projection of $u \in H$ onto the low modes by

$$P_k u = \sum_{k_0|k'| \leq k} \hat{u}_{k'} e^{ik_0 k' \cdot x}.$$

The complementary projector is denoted $Q_k = I - P_k$.

For convenience we will define the components of $u \in H$ by a range in wavenumbers

$$u_{k,k'} = (P_{k'} - P_k)u,$$

for $0 \leq k < k'$, with the convention that $u_{k,\infty} = u$ for $0 \leq k < k_0$.

The forcing

Throughout this talk we will assume that the forcing f acts only on the finite number of modes in the range of wavenumbers $[\underline{k}, \overline{k}]$. Namely,

$$f = \sum_{\underline{k} < k_0 |k| \leq \overline{k}} \hat{f}_k e^{i k_0 k \cdot x}.$$

Note that this is equivalent to

$$P_{\underline{k}} f = 0, \quad Q_{\overline{k}} f = 0.$$

Solution operators and the global attractor

Let us consider the family of solution operators $\{S(t)\}_{t\geq 0}$ that associates, to each $u^{in} \in H$, the "semi-flow" $u(t) = S(t)u^{in}$ at time $t \geq 0$. From the global existence and uniqueness of the solutions of the two-dimensional NSE we can deduce that the family of solution operators possesses a semigroup property

$$S(t) \circ S(s) = S(t+s), \quad \text{for all} \quad t, s \geq 0.$$

The global attractor for the family $\{S(t)\}_{t\geq 0}$ is defined by

$$\mathcal{A} = \bigcap_{t\geq 0} S(t)B,$$

where B is a bounded absorbing set, namely $S(t)B \subseteq B$, for all $t \geq 0$. Equivalently, $\mathcal{A}$ is the largest bounded, compact, invariant set

$$S(t)\mathcal{A} = \mathcal{A}, \quad \text{for all} \quad t \geq 0.$$

Making sense of the time averages

Definition 1. *A generalized limit is any linear functional, denoted $Lim_{T\to\infty}$, defined on the space $B([0,\infty))$ of all bounded real-valued functions on $[0,\infty)$ and satisfying*

1. $\mathrm{Lim}_{T\to\infty}\, g(T) \geq 0$, *for every $g \in B([0,\infty))$ with $g(s) \geq 0$, for all $s \geq 0$.*
2. $\mathrm{Lim}_{T\to\infty}\, g(T) = \lim_{T\to\infty} g(T)$, *for every $g \in B([0,\infty))$, whenever the usual limit exists.*

The generalized limit satisfies the following properties

1. $\liminf_{T\to\infty} g(T) \leq \mathrm{Lim}_{T\to\infty}\, g(T) \leq \limsup_{T\to\infty} g(T)$, for every $g \in B([0,\infty))$.
2. $\left| \mathrm{Lim}_{T\to\infty}\, g(T) \right| \leq \limsup_{T\to\infty} |g(T)| \leq \sup_{T\geq 0} |g(T)|$, for every $g \in B([0,\infty))$.
3. For $f \in L^\infty(0,\infty)$ and $g(T) = (1/T)\int_0^T f(t)dt$, we have

$$\mathrm{Lim}_{T\to\infty}\, g(T+\tau) = \mathrm{Lim}_{T\to\infty}\, g(T), \quad \text{for all} \quad \tau \geq 0.$$

Invariant measures

Definition 2. *A probability measure μ on H is called an invariant measure for the semigroup $\{S(t)\}_{t\geq 0}$ if*

$$\mu(E) = \mu(S^{-1}(t)E),$$

for all $t \geq 0$ and every measurable set $E \subseteq H$.

Let μ be an arbitrary invariant probability measure on $D(A)$. Then μ satisfies

$$\mu(\mathcal{A}) = 1.$$

Time-average measure

In the next Theorem, which is the consequence of the Bogolyubov-Krylov theory, we associate an invariant measure with the time average.

Theorem 1. *For every $u_0 \in D(A)$ there exists an invariant probability measure μ_{u_0} such that*

$$\mathrm{Lim}_{T\to\infty} \frac{1}{T} \int_0^T \Phi(S(\tau)u_0)d\tau = \int_{\mathcal{A}} \Phi(u)\mu_{u_0}(du),$$

for all real-valued continuous (with respect to the H-norm) functions Φ on $D(A)$.

Arbitrary invariant measure

The next result allows us to infer that any estimate valid for all measures μ_{u_0} is also valid for any arbitrary invariant probability measure.

Theorem 2. *For any invariant probability measure* μ *on* $D(A)$

$$\int_{\mathcal{A}} \left(\int_{\mathcal{A}} \Phi(u)\mu_{u_0}(du) \right) \mu(du_0) = \int_{\mathcal{A}} \Phi(u_0)\mu(du_0).$$

Proof. By Birkhoff Ergodic Theorem there exists a function Φ^*, μ-almost everywhere, s.t.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi(S(\tau)u_0)d\tau = \Phi^*(u_0),$$

and

$$\int_{\mathcal{A}} \Phi^*(u)\mu(du) = \int_{\mathcal{A}} \Phi(u)\mu(du).$$

Note, that if μ is ergodic, then Φ^* is constant μ-a.e. □

Arbitrary invariant measure

Proof. (continue)

It follows that for an arbitrary invariant measure μ, the usual time-average exists μ-almost everywhere. But the generalized limit coincides with the usual limit whenever the later exists. Therefore, $\Phi^{**} = \Phi^{*}$ μ-a.e., where

$$\Phi^{**}(u_0) = \mathrm{Lim}_{T\to\infty} \frac{1}{T} \int_0^T \Phi(S(\tau)u_0)d\tau.$$

Hence, Φ^{**} is μ-measurable, and it follows that

$$\int_{\mathcal{A}} \left(\int_{\mathcal{A}} \Phi(u)\mu_{u_0}(du) \right) \mu(du_0) = \int_{\mathcal{A}} \Phi^{**}(u_0)\mu(du_0) =$$
$$= \int_{\mathcal{A}} \Phi^{*}(u_0)\mu(du_0) = \int_{\mathcal{A}} \Phi(u_0)\mu(du_0).$$

□

Enstrophy flux

Denote $p_k = P_k u$ and $q_k = Q_k u$. Multiply both sides of the equation (1) by Ap_k

$$\frac{1}{2}\frac{d}{dt}\|p_k\|^2 + \nu|Ap_k|^2 = -(B(u,u), Ap_k) + (f, Ap_k). \tag{2}$$

Using the bilinear property of the operator B, and the fact that $u = p_k + q_k$ we get

$$(B(p_k + q_k, p_k + q_k), Ap_k) = (B(q_k, q_k), Ap_k) - (B(p_k, p_k), Aq_k),$$

where we have used the following identities valid in dimension two

$$(B(p_k, p_k), Ap_k) = 0, \tag{3}$$

and

$$(B(p_k, q_k), Ap_k) + (B(q_k, p_k), Ap_k) = -(B(p_k, p_k), Aq_k). \tag{4}$$

Enstrophy flux

Finally we get

$$\frac{1}{2}\frac{d}{dt}\|p_k\|^2 + \nu|Ap_k|^2 = (B(p_k, p_k), Aq_k) - (B(q_k, q_k), Ap_k) + (f, Ap_k) =$$
$$= -L^2(\mathfrak{E}_k^{\rightarrow} - \mathfrak{E}_k^{\leftarrow}) + (f, Ap_k),$$

where

$$\mathfrak{E}_k^{\rightarrow}(u) = -\frac{1}{L^2}(B(p_k, p_k), Aq_k),$$

the rates of enstrophy transfer or enstrophy fluxes from low to high wavenumbers, and

$$\mathfrak{E}_k^{\leftarrow}(u) = -\frac{1}{L^2}(B(q_k, q_k), Ap_k),$$

the rates of enstrophy transfer or enstrophy fluxes from high to low wavenumbers, at wavenumber k.

Enstrophy flux

Following a similar procedure, except multiplying both sides of the equation (1) by Aq_k

$$\frac{1}{2}\frac{d}{dt}\|q_k\|^2 + \nu|Aq_k|^2 = L^2\mathfrak{E}_k + (f, Aq_k) = L^2(\mathfrak{E}_k^{\rightarrow} - \mathfrak{E}_k^{\leftarrow}) + (f, Aq_k),$$

where the quantity $\mathfrak{E}_k$ denotes the net rate of enstrophy transfer (or net enstrophy flux) at the wavenumber k.

If $\mathfrak{E}_k > 0$, it means that this net transfer of enstrophy occurs from low to high wavenumbers.

If $\mathfrak{E}_k < 0$, the net transfer of enstrophy is directed from high to low wavenumbers.

Energy flux

Now we can repeat the same procedure, only multiply both sides of the equation (1) by p_k and by q_k. We obtain

$$\frac{1}{2}\frac{d}{dt}|p_k|^2 + \nu\|p_k\| = -L^2\mathfrak{e}_k + (f, p_k),$$

and

$$\frac{1}{2}\frac{d}{dt}|q_k|^2 + \nu\|q_k\| = L^2\mathfrak{e}_k + (f, q_k).$$

$\mathfrak{e}_k$ is the net rate of energy transfer (or net energy flux) at wavenumber k defined by

$$\mathfrak{e}_k = \mathfrak{e}_k^{\rightarrow} - \mathfrak{e}_k^{\leftarrow},$$

where

$$\mathfrak{e}_k^{\rightarrow}(u) = -\frac{1}{L^2}(B(p_k, p_k), q_k), \quad \text{and} \quad \mathfrak{e}_k^{\leftarrow}(u) = -\frac{1}{L^2}(B(q_k, q_k), p_k).$$

Energy and enstrophy fluxes at high wavenumbers

Theorem 3. *Suppose that* $k > \overline{k}$*. Then for any invariant probability measure* μ

$$\langle \mathfrak{E}_k \rangle = \frac{\nu}{L^2} \left\langle |AQ_k u|^2 \right\rangle ,$$

and

$$\langle \mathfrak{e}_k \rangle = \frac{\nu}{L^2} \left\langle \|Q_k u\|^2 \right\rangle ,$$

where average is taken with respect to the measure μ*.*

Energy and enstrophy fluxes at high wavenumbers

As an immediate consequence of the Theorem 3 it follows that the direction of the cascades of both energy and enstrophy is direct – from low to high wavenumbers. Moreover, at the scales smaller than that at which the energy is pumped into the system, the enstrophy cascade dominates that of the energy.

Corollary 4. *If $k > \overline{k}$, then for any invariant probability measure μ,*

$$0 \leq \langle \mathfrak{e}_k \rangle \leq \frac{1}{k^2} \langle \mathfrak{E}_k \rangle \, .$$

Energy and enstrophy fluxes at low wavenumbers

Theorem 5. *Suppose that $k \leq \underline{k}$. Then for any invariant probability measure μ*

$$\langle \mathfrak{E}_k \rangle = -\frac{\nu}{L^2} \left\langle |AP_k u|^2 \right\rangle ,$$

and

$$\langle \mathfrak{e}_k \rangle = -\frac{\nu}{L^2} \left\langle \|P_k u\|^2 \right\rangle ,$$

where average is taken with respect to the measure μ.

Energy and enstrophy fluxes at low wavenumbers

As an immediate consequence of the Theorem 5 it follows that the direction of the cascades of both energy and enstrophy is inverse – from high to low wavenumbers.

Corollary 6. *If* $k \leq \underline{k}$*, and* $k > k_0$*, then for any invariant probability measure* μ*,*

$$k^2 \langle \mathfrak{e}_k \rangle \leq \langle \mathfrak{E}_k \rangle \leq 0.$$

Kraichnan theory: Empirical assumptions

The Kraichnan theory of the fully developed turbulence in 2D is based on:

1. At length scales much smaller than those of the enstrophy feeding structures, fully developed turbulence always looks the same.
2. At the upper range (lowest wavenumbers) of the length scales in (1), eddies break up into eddies of about half their linear size while traveling a distance comparable to their linear size.
3. Most of the viscous dissipation of enstrophy takes place at the length scales in (1) which are much smaller than those in (2).
4. The range in (3) is dominated by viscous effects.
5. At the lower range (highest wavenumbers) of the length scales in (1), no significant relative movements occur.

Kraichnan spectrum for two-dimensional turbulence

We would like to consider e_k – the average energy per unit mass of the eddies of size l, where $\frac{1}{2k} \leq l < \frac{1}{k}$. In terms of the solutions of the NSE, we have

$$e_k = \frac{1}{L^2} \left\langle |u_{k,2k}|^2 \right\rangle .$$

Another important quantity is the enstrophy dissipation rate

$$\eta = \frac{\nu}{L^2} \left\langle |Au|^2 \right\rangle .$$

The mechanism in the assumption (2) can be expressed in terms of the following quantities associated with the length scale $l \sim k^{-1}$,

- U_k – average velocity of eddies of size l.
- t_k – average time for those eddies to travel the distance l.
- E_k – average enstrophy per unit mass of eddies of linear size l.

Kraichnan spectrum for two-dimensional turbulence

Then,

$$U_k \sim e_k^{1/2}, \quad t_k \sim \frac{l}{U_k} \sim \frac{1}{k e_k^{1/2}}.$$

On the other hand,

$$E_k = \frac{1}{L^2} \left\langle |A^{1/2} u_{k,2k}|^2 \right\rangle \sim k^2 e_k.$$

According to the assumption (2), the enstrophy flux per unit mass per unit time through wavenumber k,

$$\eta_k = \frac{1}{L^2} \left\langle \mathfrak{E}_k \right\rangle ,$$

accounts for most of the enstrophy dissipation of the eddies with linear size $l \sim k^{-1}$ during the characteristic time t_k and, hence, should satisfy

$$\eta_k \sim \frac{E_k}{t_k} \sim k^3 e_k^{3/2}.$$

Kraichnan spectrum for two-dimensional turbulence

That gives us

$$e_k \sim \frac{\eta_k^{2/3}}{k^2}.$$

The length scale in the assumption (2) define a wavenumber interval $[\underline{k}_i, \overline{k}_i]$, the so called inertial range. The cascade of enstrophy mechanism in (2) means that for all $\underline{k}_i \leq k \leq \overline{k}_i$

$$\eta_{\underline{k}_i} \sim \eta_k \sim \eta_{\overline{k}_i}.$$

According to (2) and (3), most of the enstrophy which is fed into the large wavenumbers is transferred through the inertial range to be dissipated by the viscosity in the dissipation range. Therefore, the transfer of enstrophy per unit time to higher modes in the inertial range should be the same as the enstrophy dissipation rate

$$\eta_k \sim \eta.$$

Kraichnan spectrum for two-dimensional turbulence

Combining all the above we conclude that

$$e_k \sim \frac{\eta^{2/3}}{k^2},$$

for $\underline{k}_i \leq k \leq \overline{k}_i$.

It is common in physical and engineering literature to assume that in the limit $k_0 \to 0$, or equivalently, $L \to \infty$, e_k could be represented as the integral of some function $\mathcal{S}(k)$, which is called the energy spectrum, over the wavenumbers

$$e_k \sim \int_k^{2k} \mathcal{S}(s) ds.$$

Therefore, in the view of the estimate on e_k in the inertial range, we obtain

$$\mathcal{S}(k) \sim \frac{\eta^{2/3}}{k^3},$$

which is the famous Kraichnan energy spectrum of the two dimensional turbulence.

Two dimensional turbulence cascade

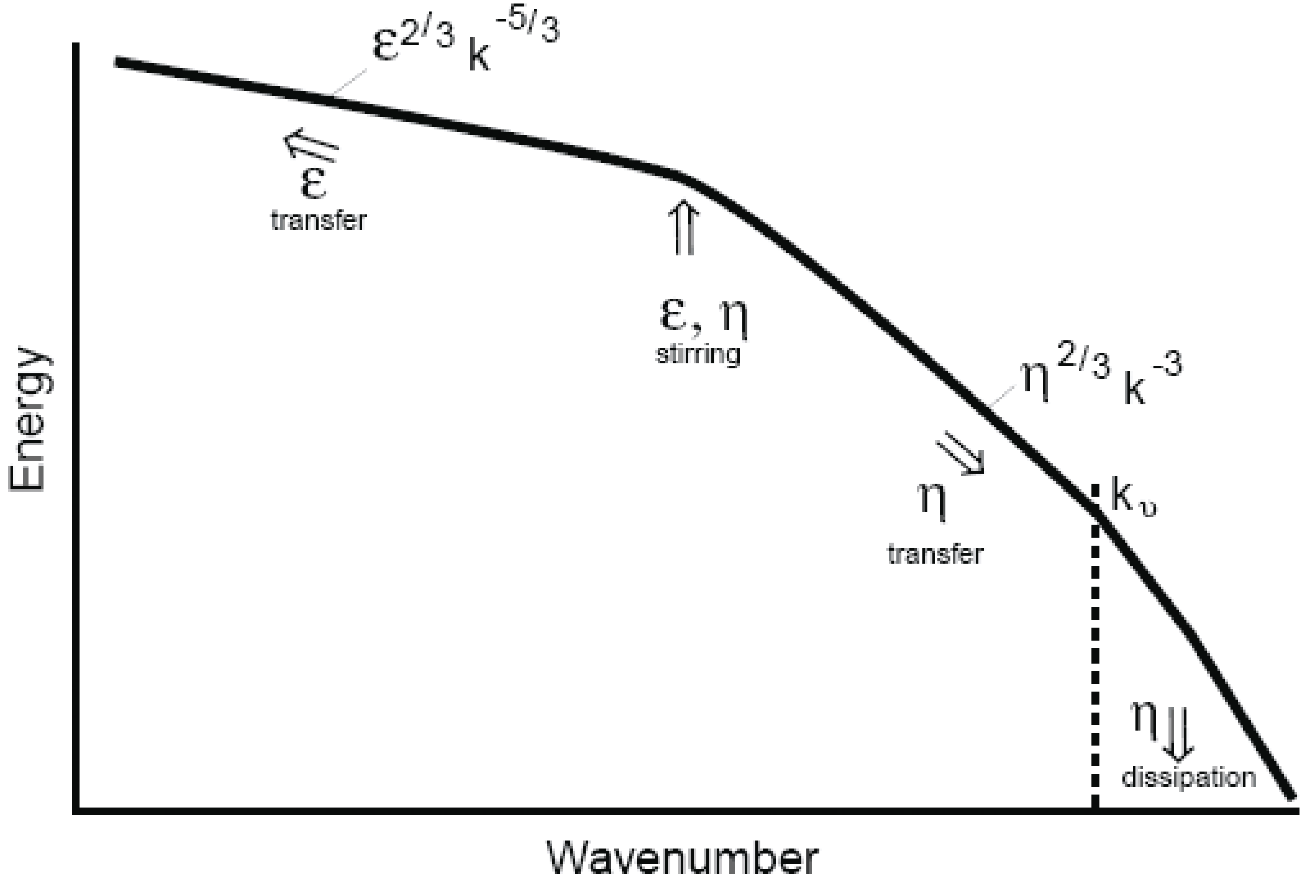

Figure 1: Two dimensional turbulence cascade.

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