Analytical and Numerical Study of Certain Models of Turbulence

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Abstract

In my thesis I study two different models of turbulence. The first part of my research concerns the, so-called, shell models of turbulence. Shell models had attracted a lot of interest as useful phenomenological models that mimics certain features of the Navier-Stokes equations (NSE). The shell models capture different aspects of the real turbulent flows, such as the energy cascade, an anomalous scaling, etc. At the same time they are an infinite set of ordinary differential equations with a simple quadratic non-linearity, which can be effectively studied numerically.

In my thesis I initiate the analytical study of shell models. I obtain some analytic results concerning the existence of unique regular solutions of the, so called, viscous Sabra shell model of turbulence and the finite dimensionality of its underlying long-term dynamics. Furthermore, I continue this line

of investigation by considering the statistical properties of the Sabra shell model. In particular, I give a rigorous proof to the recently stated conjecture, that the scaling of the structure functions of the certain linear advection model and the nonlinear field coincide.

In the second part of my research, I study the Navier-Stokes-Voigt (NSV) model of viscoelastic incompressible fluid. Recently, this model has been proposed as a regularization of the 3D Navier-Stokes equations for the purpose of direct numerical simulations. I consider the long-time dynamics of the three-dimensional NSV model and prove that the global attractor of the model, driven by an analytic forcing, consists of analytic functions. A consequence of this result is that the energy spectrum of the solutions of the 3D NSV system, that are lying on the global attractor, have exponentially decaying tail, despite the fact that the equations behave like a damped hyperbolic system, rather than the parabolic one. This result provides an additional evidence that the 3D NSV with the small regularization parameter enjoys similar statistical properties as the 3D Navier-Stokes equations.

Finally, I combine the results of the two independent lines of research. Namely, I investigate the statistical properties of the Navier-Stokes-Voigt model by employing phenomenological heuristic arguments supporting the findings by the numerical simulations of the Sabra shell model analogue of the NSV model. For large values of the regularizing parameter, compared to the Kolmogorov length scale, simulations exhibit multiscaling inertial range, and dissipation range displaying low intermittency. These facts provide evidence that the NSV regularization may reduce the stiffness of direct numerical simulations of turbulent flows, with a small impact on the energy containing scales.

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> I will greatly praise HaShem with my mouth; I will praise him among the multitude!
>
> Psalms, 109:30

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1 Introduction

The thesis contains the survey of results obtained during my research that were published in [9], [23], [24], [25], [50] and [58]. I tried to make the thesis as self-contained as possible. Nevertheless, all the details can be found in the accompanying manuscripts.

The dynamics of fluids especially erratic flows – turbulence, provides various highly challenging theoretical, as well as experimental and computational problems for engineers, physicists and mathematicians. It is widely believed that all the information about turbulence is contained in the dynamics of the solutions of the Navier-Stokes equations (NSE) for viscous, incompressible, homogenous fluids

$$\frac{\partial u}{\partial t} - \nu\triangle u + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p = f, \tag{1.1a}$$

$$\nabla\cdot u = 0, \tag{1.1b}$$

$$u(x,0) = u^{in}(x). \tag{1.1c}$$

These equations describe the evolution of the fluid velocity field $u(x,t)$ and the scalar pressure field $p(x,t)$, driven by the given external force f, with initial value $u^{in}(x)$, in 2 or 3 spatial dimensions. $\nu > 0$ represents the kinematic viscosity, and $\rho > 0$ is a constant density. The dynamics of the inviscid ($\nu = 0$) ideal fluid is governed by the Euler equations for inviscid, incompressible, homogenous flows

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p = f, \tag{1.2a}$$

$$\nabla\cdot u = 0, \tag{1.2b}$$

$$u(x,0) = u^{in}(x). \tag{1.2c}$$

Both the 3D Navier-Stokes and the Euler equations are extremely challenging to study analytically as well as computationally. The problems of the global existence and uniqueness of the solutions of the three-dimensional Navier-Stokes and Euler equations are among the most challenging problems of contemporary mathematics (see, e.g., [18], [29]). Furthermore, the real world turbulent flows possess a huge number of degrees of freedom since it involves a wide spectrum of spatial and temporal scales. Therefore, simulating them using the state-of-the-art computers is still a very difficult, and in some cases, is still an impossible task.

In my research I study two different models. The first, which is introduced in Section 2, is a representative of the family of the, so-called, shell models of turbulence. They can be viewed as simplified phenomenological models of turbulence, that capture certain phenomena of the original hydrodynamic equations – the NSE. Another model is the 3D Navier-Stokes-Voigt model of non-Newtonian viscoelastic fluid that was recently suggested in [14] as a novel inviscid regularization model for the 3D Navier-Stokes and Euler equations. The model and the motivation for investigating it will be discussed in Section 3.

2 Shell models of turbulence

2.1 Introducing the shell models of turbulence

In order to study various aspects of turbulence, and motivated by the difficulties related to the complexity of the 3D Navier-Stokes equations, simpler models and simplifications of hydrodynamics equations have been proposed over the years. Probably the first, and the most famous among them was proposed by Lorenz in

[64]. By considering the three-mode Galerkin truncation of the Boussinesq equations for fluid convection in a two-dimensional layer heated from below he obtained a remarkable evidence to the limits of predictability in weather prediction. Another successful approach in studying various aspects of a real flows by considering simplified models of hydrodynamic equations was initiated by Obukhov. In early 1970's he obtained remarkable results concerning the development of instabilities in the real fluid motion in the ellipsoid by studying the low dimensional Galerkin approximations of the Euler equations (see [40] and reference therein). Later, in the paper [70], devoted to the atmosphere studies, he proposed a simple model for studying the cascade mechanism of energy transfer in the developed turbulence. Inspired by the success of this approach he later defined the general concept of the systems of the hydrodynamic type (SHT), and advertised that the study of such systems may shed light on various problems of turbulence. All the above models, as well as the shell models of turbulence, which are the main subject of the first part of my research, are particular cases of the SHT. More information and examples can be found in [40].

The shell models of turbulence had attracted a lot of interest as a useful phenomenological model that mimics certain statistical features of the Navier-Stokes equations. The idea behind it is to replace the fluctuation of a turbulent field in an octave of wave numbers $\lambda^n < |k_n| \leq \lambda^{n+1}$ by one or a few representative variables. The range of wave numbers is called a shell, and the variables are called shell variables. Inspired by the NSE, the time evolution of the shell velocities is governed by an infinite system of coupled ordinary differential equations with quadratic nonlinearities, with forcing applied to the large scales and viscous dissipation, although acting on all scales, but becomes stronger and stronger on shells with higher wave numbers. Therefore, the shell models could be viewed as a drastic phenomenological modification of the original NSE in the Fourier space.

In my thesis I study the so called Sabra shell model of turbulence, which was introduced in [63]. However, it is worth noting, that the ideas and my analytical results that I obtained for the Sabra model, may apply equally to, for instance, the popular Gledzer-Okhitani-Yamada (GOY) shell model, introduced in [71] (see also [12], [37], [39], [40], and references therein). A recent review of the subject emphasizing, in particular, the applications of the shell models to the study of the energy-cascade mechanism in turbulence can be found in [11].

The Sabra shell model of turbulence describes the dynamics of a complex "Fourier" component of a scalar velocity field denoted by u_n. The associated one-dimensional wave number are denoted by k_n, where the discrete index n is referred as the "shell index". The equations of motion of the Sabra shell model of turbulence have the following form

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \tag{2.1a}$$

$$u_n(0) = u_n^{in}, \tag{2.1b}$$

for $n = 1, 2, 3, \ldots$, with the boundary conditions $u_{-1} = u_0 = 0$. The u_n^{in} and f_n, for $n = 1, 2, 3, \ldots$ – are given initial conditions and the forcing respectively. The wave numbers k_n are taken to be

$$k_n = k_0\lambda^n, \tag{2.2}$$

with $\lambda > 1$ being the "shell spacing" parameter, and $k_0 > 0$. Although the equation does not capture any geometry, we will consider $L = k_0^{-1}$ as a fixed typical integral length scale of the model. In an analogy to the Navier-Stokes equations $\nu > 0$ represents a kinematic viscosity and f_n, $n = 1, 2, 3, \ldots$, are the "Fourier" components of the forcing.

The three parameters of the model a, b and c are real. In order for the Sabra shell model to be a system of the hydrodynamic type we require that in the inviscid ($\nu = 0$) and unforced ($f_n = 0$, for all n) case the model should have at least one quadratic positive definite quantity to be invariant. Such a quantity will represent the kinetic energy in the system. Indeed, in order to require that the energy

$$\mathbb{E} = \sum_{n=1}^{\infty} |u_n|^2, \tag{2.3}$$

will be, at least formally, conserved we assume the following relation between the parameters of the model, which we will refer as an "energy conservation assumption"

$$a + b + c = 0. \tag{2.4}$$

Moreover, in the inviscid and unforced case the model possesses another quadratic invariant (formally)

$$\mathbb{W} = \sum_{n=1}^{\infty} \left(\frac{a}{c}\right)^n |u_n|^2. \tag{2.5}$$

For $\frac{a}{c} < 0$ this quantity is not sign definite and thus it is common to associate it with the "helicity" – in an analogy to the three-dimensional turbulence. In the case of $\frac{a}{c} > 0$ we will associate $\mathbb{W}$ with an "enstrophy" – in an analogy to the two-dimensional turbulence.

The Sabra shell model (2.1) has the following 6 parameters: ν, λ, k_0, a, b, and c. However, the "characteristic length-scale" k_0 does not appear on it's own, but only in the following combinations: $k_0 a, k_0 b$, and $k_0 c$. Therefore, without lost of generality we may assume that $k_0 = 1$. Next, by rescaling the time

$$t \to at,$$

and using the "energy conservation assumption" (2.4) we may set

$$a = 1, \quad b = -\epsilon, \quad c = \epsilon - 1. \tag{2.6}$$

Therefore, the Sabra shell model is in fact a three-parameter family of equations with parameters $\nu > 0$, ϵ, and λ. We are interested in the case were the shell sizes grow geometrically (see (2.2)), therefore we limit ourselves to $\lambda > 1$. In most of the numerical investigations of the shell models this parameter was set to $\lambda = 2$ (see, e.g., [12], [63]). The GOY shell model in the limit of $\lambda \to 1$ was first considered by Parisi in [78] and is usually referred in the literature as the *continuum shell model*, or the *zero spacing limit of the shell model* (see also [1], [2], [12]).

Considering the expression (2.5) more carefully, we see that for $|\frac{a}{c}| \le 1$, the two conserved quantities $\mathbb{E}$ and $\mathbb{W}$ become equivalent in a sense that for each value of the ratio $|\frac{a}{c}|$ there exists an absolute constant $0 < C \le 1$ such that

$$-C \cdot \mathbb{E} \le \mathbb{W} \le C \cdot \mathbb{E}.$$

Intuitively, the above means that the conservation of $\mathbb{W}$ becomes less important in our analytical estimates. Therefore, the interesting and more physically relevant behavior of the Sabra shell model is expected in the region $|\frac{a}{c}| = |\frac{1}{\epsilon-1}| > 1$, which gives us

$$0 < \epsilon < 2. \tag{2.7}$$

For the deeper physical reasoning of this choice see [63].

Now we can return to the identification of different regimes of the Sabra shell model, depending on the sign of $\mathbb{W}$. The three-dimensional parameters regime corresponds to $0 < \epsilon < 1$. In that regime we can rewrite relation (2.5) in the form

$$\mathbb{W} = \sum_{n=1}^{\infty} \left(\frac{1}{\epsilon - 1} \right)^n |u_n|^2 = \sum_{n=1}^{\infty} (-1)^n k_n^{\alpha} |u_n|^2, \tag{2.8}$$

for

$$\alpha = -\log_{\lambda} |\epsilon - 1|. \tag{2.9}$$

The two-dimensional parameters regime corresponds to $1 < \epsilon < 2$. In that case the second conserved quadratic quantity $\mathbb{W}$ is identified with "enstrophy", and we can rewrite the expression (2.5) in the form

$$\mathbb{W} = \sum_{n=1}^{\infty} k_n^{\alpha} |u_n|^2, \tag{2.10}$$

where α is also defined by the equation (2.9).

2.2 Mathematical settings of the problem

Following the usual tradition of the NSE (see [20], [84], [85]) and in order to simplify the notation we are going to write the system (2.1) in the following functional form

$$\frac{du}{dt} + \nu A u + B(u, u) = f \tag{2.11a}$$

$$u(0) = u^{in}, \tag{2.11b}$$

in a Hilbert space H and the linear operator A, as well as the bilinear operator B will be defined below. In our case, the space H will be the sequences space

ℓ^2 over the field of complex numbers $\mathbb{C}$. In other words, this is a space of the velocity configurations having the finite energy. For every $u, v \in H$, the scalar product $(\cdot, \cdot)$ and the corresponding norm $|\cdot|$ are defined as

$$(u, v) = \sum_{n=1}^{\infty} u_n v_n^*, \quad |u| = \left(\sum_{n=1}^{\infty} |u_n|^2\right)^{1/2},$$

where v_n^* denotes the complex conjugate of v_n. Therefore, the square of the H norm of the velocity field defines the energy of the system (2.3)

$$\mathbb{E} = |u|^2.$$

We denote by $\{\phi_j\}_{j=1}^{\infty}$ the standard canonical orthonormal basis of H, i.e. all the entries of ϕ_j are zero except at the place j it is equal to 1.

The linear operator $A : D(A) \to H$ is a positive definite, diagonal operator defined through its action on the elements of the canonical basis of H by

$$A\phi_j = k_j^2 \phi_j,$$

were the eigenvalues k_j^2 satisfy the equation (2.2). $D(A)$ - the domain of A is a dense subset of H. Moreover, it is a Hilbert space, when equipped with the graph norm

$$\|u\|_{D(A)} = |Au|, \quad \forall u \in D(A).$$

Using the fact that A is a positive definite operator, we can define the powers A^s of A. Namely, for every $s \in \mathbb{R}$ and for every $u = (u_1, u_2, u_3, \dots)$

$$A^s u = (k_1^{2s} u_1, k_2^{2s} u_2, k_3^{2s} u_3, \dots).$$

Furthermore, we define the spaces

$$V_s := D(A^{s/2}) = \{u = (u_1, u_2, u_3, \dots) \ : \ \sum_{j=1}^{\infty} k_j^{2s} |u_j|^2 < \infty\},$$

which are Hilbert spaces equipped with the scalar product

$$(u,v)_s = (A^{s/2}u, A^{s/2}v), \quad \forall u,v \in D(A^{s/2}),$$

and the norm $|u|_s = \sqrt{(u,u)_s}$, for every $u \in D(A^{s/2})$. Clearly

$$V_s \subseteq V_0 = H \subseteq V_{-s}, \quad \forall s > 0.$$

The case of $s = 1$ is of a special interest for us. We denote $V = D(A^{1/2})$ a Hilbert space equipped with a scalar product

$$((u,v)) = (A^{1/2}u, A^{1/2}v), \quad \forall u,v \in D(A^{1/2}).$$

We will also consider $V' = D(A^{-1/2})$ – the dual space of V. We denote by $\langle \cdot, \cdot \rangle$ the action of the functionals from V' on elements of V. Due to the Riesz representation theorem (see [81]) we have the identification $H \equiv H'$. Therefore, one can observe that the following inclusion holds

$$V \subset H \equiv H' \subset V',$$

hence the H scalar product of $f \in H$ and $u \in V$ is the same as the action of f on u as a functional in the duality between V' and V

$$\langle f, u \rangle = (f,u), \quad \forall f \in H, \ \forall u \in V.$$

Observe also that for every $u \in D(A)$ and every $v \in V$ one has

$$((u,v)) = (Au, v) = \langle Au, v \rangle .$$

Since $D(A)$ is dense in V one can extend the definition of the operator $A : V \to V'$ in such a way that

$$\langle Au, v \rangle = ((u,v)), \quad \forall u,v \in V.$$

In particular, it follows, that $\|Au\|_{V'} = \|u\|$, for every $u \in V$.

The bilinear operator $B(u, v)$ will be defined formally in the following way. Let $u, v \in H$ be of the form $u = \sum_{n=1}^{\infty} u_n \phi_n$ and $v = \sum_{n=1}^{\infty} v_n \phi_n$. Then

$$B(u,v) = -i \sum_{n=1}^{\infty} \Big(a k_{n+1} v_{n+2} u_{n+1}^* + b k_n v_{n+1} u_{n-1}^* + \\ + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} v_{n-1} u_{n-2} \Big) \phi_n, \tag{2.12}$$

where here again $u_0 = u_{-1} = v_0 = v_{-1} = 0$. It is easy to see that our definition of $B(u, v)$, together with the "energy conservation assumption" (2.4), implies that

$$B(u,u) = -i \sum_{n=1}^{\infty} \Big(a k_{n+1} u_{n+2} u_{n+1}^* + b k_n u_{n+1} u_{n-1}^* - c k_{n-1} u_{n-1} u_{n-2} \Big) \phi_n,$$

which is consistent with (2.1). In [23] (see also [24]) we have shown that indeed our definition of $B(u, v)$ makes sense as an element of H, whenever $u \in H$ and $v \in V$ or $u \in V$ and $v \in H$. For $u, v \in H$ we also show that $B(u, v)$ makes sense as an element of V'.

Now we are ready to recall our analytical results from [23] and [24] about the existence and uniqueness of solutions of the Sabra shell model of turbulence.

2.3 Well-posedness of the Sabra shell model

The viscous model – $\nu > 0$. The first result of the section (see [23], Theorem 2) states that the weak solutions to the Sabra shell model (2.1) exist globally in time. In other words, if we start with the velocity $u^{in} \in H$, then the solution in the space H exists for all times, which means that the kinetic energy of the system remains finite. More precisely,

Theorem 2.1. *Let $T > 0$, and let $f \in L^2([0,T), V')$ and $u^{in} \in H$. There exists*

$$u \in C([0,T], H) \cap L^2([0,T], V), \tag{2.13}$$

with

$$\frac{du}{dt} \in L^2([0,T], V'), \tag{2.14}$$

satisfying the weak formulation of the equation (2.1), namely

$$\left\langle \frac{du}{dt}, v \right\rangle + \nu \left\langle Au, v \right\rangle + \left\langle B(u,u), v \right\rangle = \left\langle f, v \right\rangle \tag{2.15a}$$

$$u(x,0) = u^{in}(x), \tag{2.15b}$$

for every $v \in V$, and for almost every $t \in [0,T]$.

In addition, these solutions are unique (see [23], Theorem 3), which follows from the continuity of the solution on the initial data, that is:

Theorem 2.2. *Let $u(t), v(t)$ be two different solutions to the equation (2.1) on the time interval $[0,T]$ with the corresponding initial conditions u^{in} and v^{in} in H. Then, for every $t \in [0,T]$ we have*

$$|u(t) - v(t)| \leq e^K |u^{in} - v^{in}|, \tag{2.16}$$

where

$$K = C \int_0^T \|u(s)\| ds, \tag{2.17}$$

for some constant C depending on ϵ and λ.

It is well known that the two-dimensional Navier-Stokes equations possess a unique, regular solution globally in time (e. g., [20], [84]). However, in the case of the Sabra shell model we have a slightly stronger smoothness effect of the viscosity than in the two-dimensional NSE. We have shown that if we start with a finite energy initial velocity $u^{in} \in H$, and the forcing with a finite number of modes, namely that for some $N > 0$, $f_n = 0$ for all $n \geq N$, then the solution of the equation (2.1) instantaneously becomes as smooth as possible (see [23] for the precise statement and proof of a slightly more general result). More precisely, for every $t > 0$ and the solution $u(t) = (u_1(t), u_2(t), u_3(t), \dots)$, the spectrum $|u_n(t)|$ decays to 0 exponentially with the wave number k_n (or super exponentially with n). The same result is known for the solutions of the two-dimensional NSE, however, only for the initial velocity having a finite enstrophy, or in our case, $u^{in} \in V$ (see the proof in [36], and its generalization to other types of equations in [30]).

The inviscid model – $\nu = 0$. The question of the well posedness of the Sabra shell model of turbulence in the inviscid case is more delicate. Let us consider the inviscid Sabra shell model problem

$$\frac{du}{dt} + B(u,u) = f \tag{2.18a}$$

$$u(0) = u^{in}. \tag{2.18b}$$

Recall, that the regular solutions of the Euler equations of the incompressible fluid in three-dimensional are known to exist only for a short interval of time (e.g., [7], [52], [60], [61], [67], [83]). The question of the finite time blow-up of solutions of the three-dimensional Euler equations is one of the most challenging problems of the mathematical theory of hydrodynamics today. However, the two-dimensional Euler equation possesses a global regular solution for smooth enough initial value (see, e.g., [67], [88]).

Unlike the case of the three-dimensional Euler equations, in [24] we were able to define a notion of weak solutions of the Sabra inviscid model in the following sense.

Definition 2.1. Let $0 < T < \infty$, then $u \in L^\infty([0,T],H)\bigcap C([0,T],H_w)$ is called a weak solution of the system (2.18) on the interval $[0,T]$ if for every $0 \leq t \leq T$ it satisfies

$$\langle u(t), v\rangle + \int_0^t \langle B(u(s),u(s)), v\rangle \, ds = \left\langle u^{in}, v\right\rangle + \langle f, v\rangle \,, \tag{2.19}$$

for every $v \in V$.

In [24] we proved the global in time existence of the weak solutions to the inviscid shell model (2.18).

Theorem 2.3. *Let $u^{in}, f \in H$, then for every $0 < T < \infty$ a weak solution*

$$u(t) \in L^\infty([0,T],H)\bigcap C([0,T],H_w), \tag{2.20}$$

in the sense of Definition 2.1 exists. In addition,

$$\frac{du}{dt} \in L^\infty([0,T],V_{-1}). \tag{2.21}$$

Similar result for the GOY model with the stochastic forcing were proved in [5]. The question of the uniqueness and regularity of those weak solutions still remains open in general. However, we prove the uniqueness of weak solutions in certain regularity class. We also show that every weak solution $u(t) = (u_1(t), u_2(t), \dots)$ conserve the energy provided that the components of the solution satisfy the decay estimate

$$|u_n| \leq C k_n^{-1/3} (\sqrt{n}\log(n+1))^{-1},$$

for some positive absolute constant C, namely, provided the solution is regular enough. A similar result for the solutions of Euler equations is known as the Onsager's conjecture (see [75]) and it was proved in [19] (see also [15], [27], [28]).

The following theorem, proven in [24], shows the short-time existence of a classical solutions to the inviscid shell model (2.18). Later, we show the global existence of those solutions for values of the governing parameter corresponding to the two-dimensional regime, in analogy to the two-dimensional Euler equations.

Theorem 2.4. *Let* $u^{in} \in V_d$ *and* $f \in V_d$ *for* $d \geq 1$. *There exists a time* $T_* > 0$, *depending only on the parameters of the problem and the initial conditions, such that the inviscid problem (2.18) has a unique solution*

$$u(t) \in C^1((-T_*, T_*), V_d).$$

Furthermore, we are able to derive a criterion for the blow-up of the regular solutions to the Sabra shell model. A celebrated Beale-Kato-Majda Theorem (see [8], [67]) states, citing the original article, "if a solution of the Euler or Navier-Stokes equations is initially smooth and loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches." More precisely, if the initially smooth solution of the Euler (1.2) or Navier-Stokes (1.1) equations cannot be continued to the time T_*, and T_* is the first such time, then

$$\int_0^{T_*} \|\omega(\cdot, s)\|_{L^\infty} ds = \infty,$$

where $\omega = curl(v)$ is the vorticity and v is the velocity field of Euler equations.

It means, in particular, that

$$\limsup_{t\to T_*^-} \|\omega(\cdot, s)\|_{L^\infty} = \infty.$$

Inspired by this result we have derived a similar criterion for the blow-up of solutions of the inviscid Sabra shell model (see [24], Theorem 2). In our case, the analog of the L^∞ norm of the vorticity would be the ℓ^∞ norm of the velocity "derivative"

$$\sup_{1\le n\le\infty} k_n|u_n|.$$

Clearly, if this quantity becomes infinite at some finite moment of time, then all higher order norms, namely $|u|_d$, for $d \geq 1$, become unbounded at the same time. However, in the spirit of the Beale-Kato-Majda result for the Euler equation, we show that the opposite is also true. The following theorem is essentially Theorem 6 of [24].

Theorem 2.5. *Consider the solution of the inviscid Shell model equation*

$$u \in C^1([0, T_*), V_d), \tag{2.22}$$

with initial condition $u^{in} \in V_d$*, for some* $d \geq 1$*. Suppose also that* T_* *is the maximal time of existence of the solution* $u(t) = (u_1(t), u_2(t), u_3(t), \dots)$*. Then*

$$\lim_{t\to T_*^-} \int_0^t \sup_{1\le n\le\infty} k_n|u_n(\tau)|d\tau = \infty.$$

Recall that the Sabra shell model possesses two parameters regimes corresponding to three and two-dimensional turbulence. We have already mentioned that in the two-dimensional parameters regime the inviscid Sabra shell model (in the case $f_n = 0$ for all n) possesses, along with the energy, another conserved quantity (compare with (2.10)).

$$\mathbb{W} = \sum_{n=1}^{\infty} k_n^{\alpha} |u_n|^2 = |A^{\alpha/4} u|^2 = |u|_{\alpha/2}^2,$$

for

$$\alpha = -\log_{\lambda}(\epsilon - 1), \tag{2.23}$$

and $1 < \epsilon < 2$. Therefore, the existence of the global regular solutions of the inviscid Sabra shell model in the two-dimensional parameters regime satisfying $\alpha \geq 2$ follows immediately from Theorem 2.5 and the obvious inequality

$$\sup_{1 \leq n \leq \infty} k_n |u_n| \leq \sum_{n=1}^{\infty} k_n^2 |u_n|^2,$$

for all $u = (u_1, u_2, u_3, \dots) \in V$.

Corollary 2.6. *(2-D Global Existence) Consider the inviscid Sabra shell model (2.18), where the parameters of the model are such that α defined by relation (2.23) satisfies*

$$2 \leq \alpha < \infty.$$

Then for $u^{in} \in V_d$, $d \geq \alpha$, there exists a global classical solution u to the inviscid problem (2.18) satisfying

$$u \in C^1((-\infty, \infty), V_d).$$

2.4 Finite dimensionality of the long-time behavior of the viscous Sabra shell model

Recall that the classical theory of turbulence asserts that the turbulent flows, governed by the Navier-Stokes equations have finite number of degrees of freedom

[37], [56]. Similar arguments could be equally applied to the Sabra shell model with a non-zero viscosity. In this section we present the rigorous mathematical framework, which we developed in [23], to justify the heuristical statement that the viscous Sabra shell model of turbulence has a finite number of degrees of freedom. Note, that we cannot establish such a framework for the three-dimensional Navier-Stokes equations in turbulent regime, because we do not know whether these equations possess the unique, regular solutions. Therefore, for example, the correct notion for of the global attractor for the NSE cannot be defined using traditional notions (see [33], [84], and references therein).

The first mathematical concept, which we used to establish the finite dimensional behavior of the viscous Sabra shell model, is the global attractor. The global attractor encompasses most of the possible permanent regimes of the shell model dynamics. In particular, it contains all the steady states, time periodic solutions and their unstable manifolds. It is a subset of a space H, of the states with a finite kinetic energy, which attracts all the possible trajectories of the system. Establishment of finite Hausdorff and fractal dimensionality of the global attractor implies the potential possibility of parameterizating the permanent regimes of the dynamics in terms of the finite number of parameters.

To present the results in a self contained way we will need a few definitions. Let us consider the family of solution operators $\{S(t)\}_{t\geq 0}$ that associates, to each $u^{in} \in H$, the "semi-flow" $u(t) = S(t)u^{in}$ at time $t \geq 0$. From the existence and uniqueness of the solutions of the Sabra shell model we can deduce that the family of solution operators possesses a semigroup property

$$S(t) \circ S(s) = S(t+s), \quad \text{for all} \quad t, s \geq 0.$$

Furthermore, for every $t \geq 0$ the operator $S(t) : H \to H$ is bounded and the trajectory $t \longmapsto S(t)u^{in}$ is a continuous mapping from $[0, \infty)$ to H. The global attractor for the family $\{S(t)\}_{t\geq 0}$ is a set $\mathcal{A}$ in H with the following properties

1. $\mathcal{A}$ is compact in H.
2. $\mathcal{A}$ is invariant for the semigroup. In other words $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and hence for all $t \in \mathbb{R}$.
3. $\mathcal{A}$ attracts all the bounded sets in H. That is for every bounded set B in H,

$$dist_H(S(t)B, \mathcal{A}) = \sup_{u\in B} \inf_{v\in\mathcal{A}} |S(t)u - v| \to 0, \quad \text{as} \quad t \to \infty.$$

The global attractor, whenever exists, is unique. In order to prove that the global attractor for the Sabra shell model equation exists we showed in [23] that the solutions of the Sabra shell model (we need to impose some conditions on the forcing, but for simplicity we can take it time independent satisfying $f = (f_1, f_2, \dots) \in H$) have an *"absorbing ball"* in the space of bounded enstrophy V. In other words this means that there exists a constant $C = C(|f|, \nu, \epsilon, \lambda)$, such that every solution enters the ball of radius C in V in finite time, depending on the initial condition.

$$\|S(t)u^{in}\| \leq C, \quad t \geq T(|u^{in}|, \nu, \lambda).$$

The fact that the space V is compactly embedded in H is enough for us to deduce in [23] the existence of the global attractor for the Sabra shell model equation.

Recall that $L = k_0^{-1}$ is the characteristic integral scale of the system, and, inspired by the Kolmogorov theory of turbulence we introduce the scale l_d, which represents the largest spatial scale at which the viscosity term begins to dominate over the nonlinear "inertial" term of the Shell model equation (see [23] for the precise definition of l_d). In analogy with the conventional theory of turbulence this is also the smallest scale that one needs to resolve in order to get the full resolution for "turbulent flow" associated with the Shell model system. Then we have proved in [23] that the Hausdorff and fractal dimensions of the global attractor of the Sabra shell model of turbulence, $d_H(\mathcal{A})$ and $d_F(\mathcal{A})$ respectively, satisfy

$$d_H(\mathcal{A}) \leq d_F(\mathcal{A}) \leq \log_\lambda \left(\frac{L}{l_d}\right) + \frac{1}{2}\log_\lambda \log_\lambda \Big(C_1(\lambda^2 - 1)\Big),$$

for some constant C_1 depending on ϵ and λ. In addition, let us define the, so called, "generalized Grashof number" as

$$G = \frac{|f|}{\nu^2 k_1^3}, \tag{2.24}$$

assuming the forcing f does not depend on time. Then we have shown in [23] that

$$d_H(A) \leq d_F(A) \leq \frac{1}{2}\log_\lambda G + \frac{1}{2}\log_\lambda \Big(C_1(\lambda^2 - 1)\Big). \tag{2.25}$$

Similar results are known for the Navier-Stokes equations in 2 dimensions (see, e.g, [33], [85]).

In addition, we have described the finite dimensionality of the "flow" of the Sabra shell model in a mathematically rigorous way through the concept of determining modes. We define it as a finite set of indices $\mathcal{M} \subset \mathbb{N}$, such that whenever

$$\sum_{n\in\mathcal{M}} |u_n(t) - v_n(t)|^2 \to 0, \quad \text{as } \ t \to \infty \tag{2.26}$$

it follows that

$$|u(t) - v(t)| \to 0, \quad \text{as } \ t \to \infty. \tag{2.27}$$

The number of determining modes N of the equation is the size of the smallest set of determining modes $\mathcal{M}$ satisfying the above definition. We have proved in [23] that the first N modes u_n, $1 \leq n \leq N$, are determining modes for the Sabra shell model equation provided

$$N > \frac{1}{2}\log_\lambda(C_1 G).$$

The existence of the determining modes for the Navier-Stokes equations both in two-dimensional and three-dimensional is known. However, there exists a gap between the upper bounds for the lowest number of determining modes and the dimension of the global attractor for the Navier-Stokes equations in two-dimensional both for the no-slip and periodic boundary conditions. Our upper bounds for the dimension of the global attractor and for the number of determining modes for the Sabra shell model equation coincides. Recently it was shown in [47] that similar result is true for the damped-driven NSE and the Stommel-Charney barotropic model of ocean circulation.

Calculating the lower bounds. In [25] we continued our study of the long time behavior of the Sabra shell model of turbulence. First of all, we show that the Sabra shell model of turbulence possesses an attractor of large dimension for all values of the parameter $\epsilon \in (0,2)$, $\epsilon \neq 1$. In other words we show that for every $\epsilon \neq 1$, the Hausdorff dimension of the attractor is proportional to $\log_\lambda \nu^{-1}$ for small enough viscosity ν. Therefore, we extend the results of [86] and [87] where this was shown only for $\epsilon = 1/2$, and $\epsilon = 3/2$, corresponding respectively to the purely "three" and "two-dimensional" values of parameters.

One observes that the Hausdorff and fractal dimensions of the global attractor of the evolution equation are bounded from below by the dimension of the unstable manifold of every stationary solution (see, e.g., [4], [85]). Therefore, in order to derive lower bound for the Hausdorff and fractal dimensions of the global attractor of the Sabra shell model equation, one constructs a specific stationary solution of equation (2.11) and counts the number of linearly unstable directions of that equilibrium. The same technique was first used in [69] (see also [4], [62]) to obtain lower bounds for the dimension of the Navier-Stokes global attractor in 2D.

In our case, we introduced special kind of forcing, which one of the referees proposed to call "lacunary power-law" forcing, and showed that for that choice, the Sabra shell model of turbulence possesses an attractor of large dimension for all values of the parameter $\epsilon \in (0, 2)$, $\epsilon \neq 1$. Namely, we showed that there exist positive constants c_1, c_2, depending on the parameters of the problem and on ϵ, such that

$$d_F(\mathcal{A}) \geq d_H(\mathcal{A}) \geq c_1 \log_\lambda G + c_2. \tag{2.28}$$

Furthermore, in [25] we study the linear stability of the stationary solution of the Sabra shell model, concentrated on a single mode N. We show that it becomes unstable for every N and for small enough viscosity for all $\epsilon \in (0, 2)$, $\epsilon \neq 1$, thus correcting the result of [54]. By considering a stationary solution concentrated on an infinite number of shells, we are able to demonstrate exactly how the transition to chaos occurs both in the "two" and "three-dimensional" parameters regime, through successive bifurcations. In the "three-dimensional" regime $\epsilon \in (0, 1)$, when the parameter ϵ becomes close to 0 or 1 the scenario of the transition to chaos is different then in the rest of the interval. Namely, for a fixed viscosity, when ϵ crosses the values 0.05 and 0.97, the number of unstable directions drops by the factor of 3. However, the attractor in those regimes is still of the size proportional to $\log_\lambda \nu^{-1}$, the chaotic behavior in the vicinity of the stationary solution changes dramatically.

Moreover, in [25] we show that in the "two-dimensional" parameters regime the Sabra shell model has a trivial attractor that is reduced to a single equilibrium solution for any value of viscosity ν, when the forcing is applied only to the first shell. This result is true also for the two-dimensional NSE due to Marchioro [68].

Existence of an inertial manifold. Finally, in contrast with the present knowledge for the Navier-Stokes equations, we show that the Sabra shell model equation (2.1) possesses an inertial manifold $\mathcal{M}$. Inertial manifolds are finite dimensional Lipschitz globally invariant manifolds that attract all bounded sets in the phase space at an exponential rate and, in particular, contain the global attractor. We show that $\mathcal{M} = graph(\Phi)$, where Φ is a C^1 function which slaves the components u_n, for all $n \geq N$, as a function of $\{u_k\}_{k=1}^N$, for N large enough depending on the physical parameters of the problem, i.e. ν, f, λ, a, b, and c. The reduction of the system (2.1) to the manifold $\mathcal{M}$ yields a finite dimensional system of ordinary differential equations. This is the ultimate and best notion of system reduction that one could hope for. In other words, an inertial manifold is an exact rule for parameterizing the large modes (infinite many of them) in terms of the low ones (finitely many of them). The concept of inertial manifold for nonlinear evolution equations was first introduced in [34] (see also [21], [22], [35], [82], [85]).

2.5 Calculating scaling exponents of the Sabra shell model

One of the mysteries of the statistical theory of turbulence is related to the so called "anomalous scaling" of the structure functions. Let us define the n-th order structure function of the flow $v(\mathbf{x}, t)$, governed by the Navier-Stokes equations (1.1).

$$S_n(r) = \left\langle \left((v(\mathbf{r}, t) - v(0, t)) \cdot \frac{\mathbf{r}}{r} \right)^n \right\rangle, \tag{2.29}$$

where $\langle \cdot \rangle$ denotes ensemble or time average, and $r = |\mathbf{r}|$. Under various assumptions on the flow, and in particular, assuming that the mean energy dissipation rate $\epsilon = \lim_{\nu \to 0} \nu \left\langle (\nabla v)^2 \right\rangle$ is bounded away from zero, Kolmogorov derived the famous $4/5$-th law

$$S_3(r) = -\frac{4}{5} \epsilon r, \tag{2.30}$$

(see [37], [56], etc.). The relation (2.30) implies that the third-order structure function is universal and does not depend on the details of the turbulence production, and is determined only by the mean dissipation rate of the energy. Applying dimensional arguments, Kolmogorov further conjectured that the rest of the structure functions are also universal and scale like $S_n(r) \sim r^{n/3}$. This conjecture apparently turned out to be false (for the phenomenological arguments see [56]). Recent experiments, both numerical and laboratory, predict that the structure functions are indeed universal and for each $n \geq 2$ there exist "scaling exponents" η_n, such that for large Reynolds number $S_n(r) \sim r^{\eta_n}$. Moreover, $\eta_3 = 1$, as predicted by (2.30), but the rest of the scaling exponents differ from the normal Kolmogorov's scaling $n/3$. This phenomenon is refereed to as the "anomalous scaling" of structure functions and it's nature is still one of the most challenging problems of the nowadays statistical fluid mechanics.

As we already pointed out in the introduction, the Navier-Stokes equations provide a great challenge both for computing the high Reynolds number flows and for studying this problem analytically. However, shell models of turbulence, both GOY and Sabra, also display an anomalous scaling, which for some values of the parameter ϵ and λ is very close to that of the real-world turbulent flow. Therefore, understanding the mechanism the anomalous scaling in the linear and nonlinear shell models attracted attention of both physical and mathematical communities. In recent years a major breakthrough has been made in understanding the mechanism of anomalous scaling in the linear models of passive scalar advection. (However, we would like to note that this understanding still lacks a rigorous mathematical framework).

In a recent work [3] further insight to the anomaly of the exponents of the nonlinear problem (for the field u_n) was sought by relating them to the scaling exponents of a *linear* model for a field w_n. To connect the linear model to the nonlinear problem one considers the system of two coupled equations

$$\begin{aligned} \frac{du}{dt} &= B(u,u) + \theta B(w,u) + \nu Au + f \ , \\ \frac{dw}{dt} &= B(u,w) + \theta B(w,w) + \nu Aw + \tilde{f} \end{aligned} \tag{2.31}$$

with $\theta \geq 0$ being a real parameter and f and $\tilde{f}$ being different realizations of the same random force. Observe that for any $\theta \neq 0$, the two equations in (2.31) exchange roles under the change $\theta w \leftrightarrow u$. Thus if the scaling exponents ξ_p and ζ_p of the two fields exist (i.e. a true scaling range exists), they must be the same for all $\theta \neq 0$. For $\theta = 0$ we recover the equations for the nonlinear and the linear models, and in [3] it was *assumed* that the scaling exponents of either field exhibits no jump in the limit $\theta \to 0$. In [9] we closed this gap. We proved that the solutions of the system (2.31) exist globally in time and depend continuously on the parameter θ, including the limit $\theta \to 0$. In particular we showed that *if* the structure function of u_n and w_n exhibit the same scaling exponents for any $\theta \neq 0$, they will also have the same scaling exponents in the limit $\theta \to 0$ (including $\theta = 0$). Our rigorous results correspond to the deterministic force f and $\tilde{f}$ and could be generalized to the case of stochastic forcing.

3 The 3D Navier-Stokes-Voigt model

3.1 A regularization model for the 3D Navier-Stokes equations

In the second part of my thesis I consider the Navier-Stokes-Voigt (NSV) model of viscoelastic fluid which is obtained from the Navier-Stokes equations (1.1) by subtracting the term $\alpha^2 \triangle u_t$ from the left hand side. It is governed by the system of equations

$$u_t - \nu \triangle u - \alpha^2 \triangle u_t + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p = f, \tag{3.1a}$$

$$\nabla \cdot u = 0, \tag{3.1b}$$

$$u(x, 0) = u^{in}(x), \tag{3.1c}$$

in $\Omega \subset \mathbb{R}^3$, equipped with the periodic or Dirichlet boundary conditions. $u(x,t)$ represents the velocity field, p is the pressure, $\nu > 0$ stands for kinematic viscosity, $\rho > 0$ is a constant density, f is the forcing, and finally, α is a real positive length scale parameter, for which the ratio $\frac{\alpha^2}{\nu}$ characterizes the response time that is required for the viscoelastic fluid to respond to the applied force. The system (3.1) was first analytically studied by Oskolkov, who introduced the NSV equations (see [76], [77]) as a model of motion of linear, viscoelastic fluid.

Recently, in [14], the 3D Navier-Stokes-Voigt equations were suggested as a regularization model for the 3D Navier-Stokes equations, where α is considered a small regularization parameter. First, it was recognized that the inviscid ($\nu = 0$) version of the NSV system (3.1) coincides with the inviscid simplified Bardina sub-grid scale model of turbulence. The viscous simplified Bardina model was introduced and studied in [57] and [10] (see also [6] for the original Bardina model). In [14] the viscous and inviscid simplified Bardina models were shown to be globally well-posed. Moreover, it was also shown that the viscous simplified Bardina model has a finite dimensional global attractor, and its dimension was estimated in terms of the physical parameters. The energy spectrum of the viscous simplified Bardina model was investigated in [14]. Viewed from the numerical analysis point of view the authors of [14] proposed the inviscid simplified Bardina model (or equivalently the inviscid NSV equations) as an inviscid regularization (because no additional viscosity or hyperviscosity are introduced) of the 3D Euler equations, subject to periodic boundary conditions. Motivated by this observation system (3.1) was also proposed in [14] as a regularization, of the 3D Navier-Stokes equations for the purpose of direct numerical simulations for both the periodic and the no-slip Dirichlet boundary conditions. Notice that the NSV equations have an important advantage comparing to other regularization models used in ocean dynamics, like hyperviscosity or α-models. This model does not require additional artificial boundary conditions, which cause difficulties and possibly exhibit non-physical behavior in applications, such as non-physical boundary layer.

The addition of the $-\alpha^2\triangle u_t$ term has two main effects. On one hand, it regularizes the equation in a way that the three-dimensional system (3.1) becomes now globally well-posed (see [14], [76]). On the other hand, as was noted in [51], it changes the parabolic character of the original Navier-Stokes equations. In particular, one does not observe any immediate smoothing of the solutions, unlike what is expected in parabolic PDEs. We also remark that this type of inviscid regularization has been recently used for the two-dimensional surface quasi-geostropic model [53]. Most importantly, necessary and sufficient conditions for the formation of singularity were presented in terms of the regularizing parameter. Finally, the NSV model (3.1) has the same stationary solutions as the Navier-Stokes equations (1.1), and the averaged solutions of both the NSV model and the NSE satisfy the same Reynolds Averaged Equations, although the dynamics of the two equations is expected to be different. Those facts suggest that there exists a strong link between the statistical properties of the NSV model and the statistical properties of turbulent flows.

3.2 On the analyticity of global attractor of the 3D NSV model

The long-time dynamics of system (3.1), in $\Omega \subset \mathbb{R}^3$, equipped with the periodic boundary conditions, has been studied in [49] and [51]. First, the existence of the finite-dimensional global attractor of the system has been established. Furthermore, upper bounds for the number of determining modes, and the fractal dimension of the global attractor of the 3D NSV model where derived in [51]. In particular, it was shown that the attractor lies in the bounded subset of the Sobolev space $H^1(\Omega)$, whenever the forcing term $f \in L^2(\Omega)$. In [79] the authors proved the convergence of the Statistical Stationary solutions (SSS) of the 3D NSV to corresponding SSS of the 3D NSE as $\alpha \to 0$, showing that for a small α, the statistics of the NSV and NSE should be practically the same.

The main goal of the research published in [50] was to provide further support for the proposal made in [14] that the NSV system (3.1), with the small regularization parameter α, can be used as a numerical model for studying the original Navier-Stokes equations, and in particular their statistical properties. Indeed, in [50] we show that the global attractor of the 3D NSV model consists of the real analytic functions, whenever the forcing term f is analytic. The idea is to construct an asymptotic approximation $v(x,t)$ to the solution $u(x,t)$ of the system (3.1) satisfying

$$\lim_{t\to\infty} \|v(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} = 0,$$

and show that $v(x,t)$ is a real analytic function of space. To show that we used the concept of the Gevrey class regularity. For a given $\tau > 0$, and $r \geq 0$, we define the following family of Gevrey spaces

$$G^r_\tau(\Omega) := \{u \in L^2(\Omega) \ : \ \sum_{j\in\mathbb{Z}^3} |u_j|^2 |j|^{2r} e^{2\tau|j|} < \infty\}.$$

One can show that the space of real analytic functions $C^\omega(\Omega)$ has the following characterization

$$C^\omega(\Omega) = \bigcup_{\tau>0} G^r_\tau(\Omega),$$

for any $r \geq 0$ (see, e.g., [59]). The concept of the Gevrey class regularity for showing the analyticity of the solutions of the Navier-Stokes equations, was first introduced in [36], simplifying earlier proofs. Later this technique was extended to the large class of analytic nonlinear parabolic equations in [30].

Functions belonging to this Gevrey regularity class are characterized by the exponential decay of the tail of their Fourier coefficients. Therefore, we actually proved that the global attractor of the 3D NSV system driven by the smooth enough forcing, consists of real analytic functions $u(x,t)$, whose Fourier spectrum $\hat{u}(k,t)$ satisfies the decay estimate

$$|\hat{u}(k,t)| \leq c|k|^{-2}e^{-|k|/\lambda^{1/2}}, \tag{3.2}$$

for some λ, depending on the parameters of the problem. Our method of the proof – splitting of $v(x,t)$ into higher and lower Fourier components, has been used before in the context of the weakly damped driven nonlinear Schrödinger equation in [72] and a model of Bénard convection in a porous medium in [73] (see also [41]).

One important consequence of our result is that the solutions of the 3D NSV system (3.1) lying on the global attractor posses a dissipation range, despite the fact that the equations behave like the damped hyperbolic system, rather than the parabolic equation. This fact provides an additional evidence that (3.1), with the small regularization parameter α, can indeed be used as a model to study the statistical properties of turbulent solutions of the 3D Navier-Stokes equations, a subject of ongoing research.

Finally, we obtain bounds for the exponential decaying length scale of the general solutions of the NSV system lying on the attractor. The obtained estimate is similar to the bounds for the smallest length scale in the turbulent flow that was previously calculated for the solutions of the 3D Navier-Stokes equations in [26]. Using the inequality (3.2), and following the ideas of [26] (see also [44], [45] for a different approach), the quantity $1/\lambda^{1/2}$, can be naturally identified as the *exponential decaying length scale*, since the exponential decay of the spectrum of u

is effective only at high wavenumbers satisfying $|k| > \lambda^{1/2}$. This fact allowed us to estimate the exponential decaying length scale of the NSV equations by providing lower bounds for λ. We proved that this estimate has the same asymptotic behavior as the estimate of the characteristic length scale of the 3D Navier-Stokes equations obtained in [26], without requiring any additional assumptions on the regularity of the flow of the system (3.1).

In addition, using the techniques introduced in [74], we estimate the exponential decaying scale of the stationary solutions of the 3D NSV and 3D Navier-Stokes equations. Our bounds coincide with those obtained in this paper for the general solutions of the NSV system lying on the global attractor, and for those of the 3D Navier-Stokes equations reported in [26].

3.3 The statistical properties of the 3D NSV model

In [58] we investigate both analytically and numerically the effect of the Navier-Stokes-Voigt (NSV) type regularization on the spectra of the structure functions of the Navier-Stokes equations (NSE). First of all, it is easy to see from (3.1) that besides having the same steady state solutions as the NSE, the NSV model satisfies formally the same infinite time Reynolds averaged equations as those for the NSE, suggesting a strong link with the statistical properties of turbulent flows. However, as it was observed above, the NSV model presents an extra length scale associated to the viscoelasticity, the parameter α, besides the well known Kolmogorov length scale, η, which is usually associated to the smallest scales of motion in turbulent flows. Following the Kolmogorov-type dimensional arguments, we have defined η^{NSV} – the smallest scale of motion of the NSV model and established the following relation

$$\frac{\eta^{NSV}}{\eta} \sim \left(\frac{\alpha}{\eta}\right)^{1/3}. \tag{3.3}$$

Equation (3.3) shows that if we choose $\eta \ll \alpha$, the degrees of freedom of the NSV flow are significantly reduced in comparison to the usual Navier-Stokes flow, which provides a great advantage concerning direct numerical simulations. In Figure 1 one can see that this fact is supported by the simulations of the Sabra-NSV model.

Next, we consider the energy transfer mechanism of the NSV equations. Note that the NSV model satisfies the following energy equation for every $t \in [0, \infty)$,

$$\frac{d}{dt}(\frac{1}{2}\left|\mathbf{u}(\cdot,t)\right|^2 + \frac{\alpha^2}{2}\left|\nabla\mathbf{u}(\cdot,t)\right|^2) = (\mathbf{f}, \mathbf{u}(\cdot,t)) - \nu\left|\nabla\mathbf{u}(\cdot,t)\right|^2. \tag{3.4}$$

Therefore, a positive quadratic conserved quantity in the inviscid, $\nu = 0$, unforced, $\mathbf{f} = 0$, and periodic or no-slip setting, which was proved rigorously in [14], is

$$S_2^\alpha = \frac{1}{2}\left|\mathbf{u}\right|_{L^2}^2 + \frac{\alpha^2}{2}\left|\nabla\mathbf{u}\right|^2, \tag{3.5}$$

which we call the α-energy. The quantity

$$S_2 = \frac{1}{2}\left|\mathbf{u}\right|^2 \tag{3.6}$$

is the usual kinetic energy, and we remark that it is not conserved for the inviscid unforced NSV equations. Because the kinetic energy is not a conserved quantity, the arguments used by Kraichnan in [55] (see also [31]) to study the turbulent cascade scenario cannot be employed directly to the kinetic energy. However, we carried out all the calculations for the conserved α-energy, S_2^α, (3.5), and recovered the conclusions for the kinetic energy, S_2, (3.6). The similar strategy was also used in [13, 14, 16, 32, 42, 65, 66] for studying various α subgrid scale models of turbulence.

Furthermore, we study the energy distribution scale-by-scale for the 3D NSV equations. Let $\langle \cdot \rangle$ denote average with respect to an invariant measure, μ^α, for the NSV semigroup (such a measure is known to exist for the NSV, see [79]). For the Navier-Stokes equations, assuming that there exists an extensive range of wavenumbers, where the viscous dissipation does not play a significant role, one can show that the energy simply cascades through these length scales, with rate equals to the mean energy dissipation rate for the NSE – $\epsilon = \nu \langle |\nabla \mathbf{u}| \rangle$. For the NSV equations, a similar scenario holds for the α-energy (3.5) and we obtain

$$S_2^\alpha(k) \sim \epsilon_\alpha^{2/3} k^{-2/3} (1 + \alpha^2 k^2)^{1/3}.$$

Therefore, for $k \ll \alpha^{-1}$, we have a $k^{-2/3}$ range, while that for $\alpha \approx k^{-1}$, we have a power zero range. This scenario was numerically observed in the shell model simulations (see Figure 1). We remark that this scenario of two power laws in the inertial range was first proposed in [32] for the NS-α model, and then for the rest of the α models in [13, 14, 16, 46, 65, 66].

To support our theoretical arguments, we perform the computational study of the Sabra shell model equations, equipped with the NSV-type regularizetion. We use the Sabra shell model as a test bed for this idea since it is much more accessible computationally then the original NSV model (3.1). We study the following regularized version of the Sabra shell model (2.1)

$$\frac{du_n}{dt} = \frac{i(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2})}{1+\alpha^2 k_n^2} - \\ - \frac{\nu k_n^2}{1+\alpha^2 k_n^2} u_n + \frac{f_n}{1+\alpha^2 k_n^2}, \tag{3.7a}$$

$$u_n(0) = u_n^{in}, \tag{3.7b}$$

for $n = 1, 2, 3, \dots$, with the boundary conditions $u_{-1} = u_0 = 0$. Note that this regularization, namely dividing the right hand side of the equation (2.1) by $(1 + \alpha^2 k_n^2)$, is in the spirit of adding the Navier-Stokes-Voigt type regularization $-\alpha^2 \triangle u$ to the original NSE (1.1). All the parameters are the same as in the shell model (2.1), and the regularizing parameter α has a dimension of the length, i.e. of k_0^{-1}. The modified model in the inviscid ($\nu = 0$) and unforced ($f_n = 0$, for all $n = 1, 2, 3, \dots$) case conserves (formally) the following two quadratic quantities – the modified energy

$$\mathbb{E}_\alpha = \sum_{n=1}^{\infty} (1 + \alpha^2 k_n^2)|u_n|^2, \tag{3.8}$$

and the modified "enstrophy"

$$\mathbb{W}_\alpha = \sum_{n=1}^{\infty} (1 + \alpha^2 k_n^2)\left(\frac{a}{c}\right)^n |u_n|^2. \tag{3.9}$$

Considering the (3.7) we observe that the energy dissipative timescale, $(1 + \alpha^2 k_n^2)/\nu k_n^2$, converges to a constant value, the relaxation time parameter, α^2/ν, as n increases. This should be compared to the Sabra shell model case (2.1), which corresponds to the NSE, where the dissipative timescale decreases as $1/(\nu k_n^2)$, which makes DNS of the Sabra shell model, and similarly of the turbulent flows, a very stiff problem. Thus, for DNS containing a large amount of scales satisfying $k_n^{-1} < \alpha$, we expect the regularized Sabra shell model, and the corresponding full

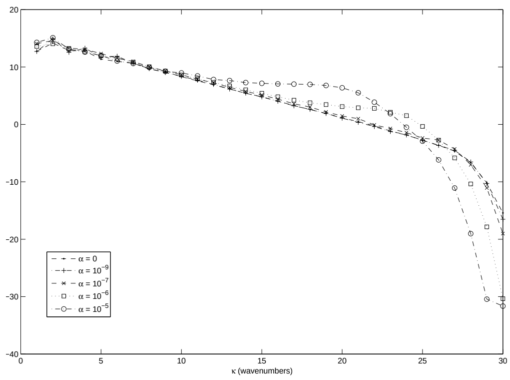

Figure 1: Averaged scale-by-scale energy spectra of the Sabra shell model with the NSV-type regularization (3.7), with $\epsilon = 1/2$, $\nu = 10^{-9}$, and different values of the parameter α. One observes that for large values of α we clearly see the appearance of the second power-law (3.11) at the large wavenumbers, while the energy spectra at the small wavenumbers still obeys the "Kolmogorov"-like power-law (3.10).

NSV model to substantially decrease the stiffness of the algorithm. On the other hand, we expect that the statistics of the regularized Sabra shell model (3.7) to be slightly different from the Sabra shell model (2.1) for spacial scales smaller than α. However, we expect to be able to tune the regularizing parameter α, namely, to make it sufficiently small, in such a way that the statistics of the energy containing scales – the inertial range – will be the same both for the Sabra model and it's regularized analog, while reducing the stiffness of the algorithm. Right now our preliminary numerical simulations support this claim.

The numerical results (see Figure 1), published in [58], supported our theoretical conclusions. Namely, for $\alpha^2 \ll k_n^{-2}$ the spectra of the regularized model is the same as the one for $\alpha = 0$. For bigger n, we observe that the statistics changes. More precisely, we start to observe the second power law of the energy spectrum. The energy at the small wavenumbers exhibit a usual "Kolmogorov"-like power-law behavior of

$$|u_n|^2 \sim k_n^{-1/3}, \tag{3.10}$$

while for larger wavenumbers it scales like a constant

$$|u_n|^2 \sim Const. \tag{3.11}$$

It is interesting to point out that although the regularized model (3.7) is not parabolic, in the long time average, the inertial range is followed by the exponential decay of the spectrum, supporting the analytical results of [50].

4 Conclusions

In the first part of my research I have initialized the rigorous analytical study of the shell models of turbulence. I have established new results concerning the existence and uniqueness of the solutions of the shell models both in the viscous and the inviscid regime. Moreover, I have studied the finite dimensionality of the long-time behavior of the models, connecting it to the known numerical investigations. In addition, I have shown that the shell model possess a finite dimensional inertial manifold. That is, there exists an exact rule which parameterizes the small scales as a function of the large scales. The existence of inertial manifolds is not known for the Navier-Stokes equations not even in the two-dimensional case.

In the second part of my research, I continued the investigation of the Navier-Stokes-Voigt model of the viscoelastic fluid. Together with the collaborators, we proved that the elements of the global attractor of the 3D NSV equations (3.1) with periodic boundary conditions, driven by an analytic forcing, are analytic. An important consequence of our result is that the solutions of the 3D NSV system (3.1) lying on the global attractor posses a dissipation range – an exponentially decaying spectrum. This fact provides an additional evidence that (3.1) with the small regularization parameter α enjoys similar statistical properties of the 3D Navier-Stokes equations, a fact that was first suggested in [14].

Finally, we investigated the statistical properties of the NSV model both by employing phenomenological heuristic arguments and performing numerical simulations of the Sabra shell model analogue of the NSV model. We found that for large values of the regularizing parameter, compared to the Kolmogorov length scale, simulations exhibit multiscaling inertial range, and dissipation range displaying low intermittency. These facts provide evidence that the NSV regularization may reduce the stiffness of direct numerical simulations of turbulent flows, with a small impact on the energy containing scales.

5 Statement about independent collaboration

This section is required by the Feinberg Graduate school as part of the thesis in the "Published Papers" format, while some the papers were done with other collaborators, to summarize my own independent efforts.

First papers [23], [24] were done and written by me, while the general idea of the analytical study of the shell models by tools known from the theory of the Navier-Stokes equations, was suggested by my advisor Prof. E. S. Titi and Prof. P. Constantin. The main idea of [25] was mine and I wrote the paper, using the help and important remarks by my advisor Prof. E. S. Titi and Prof. P. Constantin.

The main idea of the paper [9] was suggested by Prof. I. Procaccia. The rigorous mathematical statement and the analytical study were done by me together with my advisor Prof. E. S. Titi. This is a joint collaborative project with the tools developed by us in [23], [24].

Furthermore, the paper [50] was written by me, while the main idea of using the Gevrey spaces technique was suggested by my advisor Prof. E. S. Titi. Finally, in [58] my main contribution was to provide the numerical simulations of the Sabra shell model in the context of the inviscid regularization using the Navier-Stokes-Voigt idea. In addition the analytical part of the paper was a collaborative effort with Dr. F. Ramos.

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