Topics in the Local Theory of Normed Spaces
Lecture 10

1 K-convexity

Definition 1.1. *Consider the finite probability space $\Omega = \{-1,1\}^n$, equipped with the measure $\mu(A) = \frac{\#A}{2^n}$. Define the sequence of Rademacher functions to be*

$$r_i : \Omega \to \{0,1\} \quad r_i(t) = t_i.$$

Theorem 1.2. *(Khinchine's inequality) For every $1 \leq p < \infty$ there exist absolute constants $0 < C < K_p < \infty$ such that*

$$C(\sum_{i=1}^n a_i^2)^{1/2} \leq (\mathbb{E}|\sum_{i=1}^n a_i r_i(t)|^p)^{1/p} \leq K_p(\sum_{i=1}^n a_i^2)^{1/2}.$$

Proof. In the case of $p = 2$ the theorem becomes an equality as a consequence of the parallelogram law:

$$\mathbb{E}|\sum_{i=1}^n a_i r_i(t)|^2 = \frac{1}{2^n}\sum_{t\in\Omega}|\sum_{i=1}^n a_i r_i(t)|^2 = \sum_{i=1}^n a_i^2.$$

Suppose, $p > 2$ and assume that $\sum_{i=1}^n a_i^2 = 1$. According to Bernstein's inequality, (see lecture 8)

$$\mathbf{P}(|\sum_{i=1}^n a_i r_i(t)| > s) \leq 2e^{-s^2/2}.$$

By definition of the expectation and applying integration by parts we have

$$\mathbb{E}|\sum_{i=1}^n a_i r_i(t)|^p = \int_0^\infty s^p d\mathbf{P}(|\sum_{i=1}^n a_i r_i(t)| \leq s) = \int_0^\infty \mathbf{P}(|\sum_{i=1}^n a_i r_i(t)| > s) ds^p \leq$$
$$\leq 2p\int_0^\infty s^{p-1}e^{-s^2/2}ds =^{def} K_p^p.$$

One can show (by successive integration by parts of the last expression) that $K_p \leq c\sqrt{p}$ for some absolute constant $c > 0$. Moreover, Holder's inequality implies $(\mathbb{E}|f|^2)^{1/2} \leq (\mathbb{E}|f|^p)^{1/p}$ for $p \geq 2$. Hence we proved that for $2 \leq p \leq \infty$

$$(\sum_{i=1}^{n} a_i^2)^{1/2} = (\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^2)^{1/2} \leq (\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^p)^{1/p} \leq K_p(\sum_{i=1}^{n} a_i^2)^{1/2}.$$

We now turn to the case of $1 \leq p < 2$. Combining the previous case and Holder's inequality we get

$$(\sum_{i=1}^{n} a_i^2) = \mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^2 = \mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^{1/2+3/2} \leq (\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|)^{1/2}(\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^3)^{1/2}.$$

Using the last inequality and the estimation for the case $p \geq 2$ we conclude the theorem from the case $p = 1$

$$(\sum_{i=1}^{n} a_i^2) \leq K_3^{3/2}(\sum_{i=1}^{n} a_i^2)^{3/4}(\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|)^{1/2}$$

Again using Holder's inequality we get finally for $1 \leq p < 2$

$$K_3^{-3}(\sum_{i=1}^{n} a_i^2)^{1/2} \leq \mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)| \leq (\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^p)^{1/p} \leq (\mathbb{E}|\sum_{i=1}^{n} a_i r_i(t)|^2)^{1/2} = (\sum_{i=1}^{n} a_i^2)^{1/2}.$$

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Remark: Note that we have an estimate on the constant $K_p \leq c\sqrt{p}$.

We can complete the sequence of Rademacher's functions into an orthonormal basis by the following construction. For each $A \subseteq \{1, \dots, n\}$ define the Walsh function

$$W_A(t) = \prod_{i \in A} r_i(t),$$

and $W_\emptyset(t) \equiv 1$. The following are properties of the Walsh's functions

1. $|W_A| = 1$ for all $A \subseteq \{1, \dots, n\}$.

2. For every $A \neq B$ the functions W_A, W_B are orthogonal:

$$\mathbb{E}W_A W_B = \mathbb{E}\prod_{i \in A} r_i(t) \prod_{i \in B} r_i(t) = \prod_{i \in A \triangle B} r_i(t) = 0.$$

The Walsh's functions form an orthonormal system of 2^n functions in $L_2(\Omega_n)$, which has dimension 2^n, and hence is complete. Every $f \in L_2(\Omega_n)$ can be represented as

$$f(t) = \sum_{A \subseteq \{1,\dots,n\}} a_A W_A(t), \text{ and } \|f\|_{L^2(\Omega_n)}.$$

In addition, $a_A = \mathbb{E} f W_A$ and $\|f\|_{L_2(\Omega_n)} = (\sum_{A \subseteq \{1,\dots,n\}} |a_A|^2)^{1/2}$.

Another way to look at the Walsh's functions is the following. Define for each $n = 1, 2, \dots$ a square $2^n \times 2^n$ matrix W_n, where the columns represent a variable t and rows represent a subset $A \subseteq \{1, \dots, n\}$.

$$W_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and inductively

$$W_n = \begin{pmatrix} W_{n-1} & W_{n-1} \\ W_{n-1} & -W_{n-1} \end{pmatrix}$$

Definition 1.3. *Let X be a finite dimensional linear normed space. Define*

$$L_2(\Omega_n, X) = \{f : \Omega_n \to X \ : \ \|f\|_{L_2(\Omega_n, X)} = (\mathbb{E}\|f\|_X^2)^{1/2} < \infty\}.$$

In general $L_2(\Omega_n, X)$ is Hilbert space if and only if X is Hilbert. Still every $f \in L_2(\Omega_n, X)$ can be represented as $f = \sum_{A \subseteq \{1,\dots,n\}} x_A W_A(t)$, where

$$x_A = \mathbb{E} f(t) W_A(t) = \frac{1}{2^n} \sum_{t \in \Omega_n} f(t) W_A(t).$$

Definition 1.4. *Consider projection of $L_2(\Omega_n, X)$ onto the subspace spanned by the n Rademacher's functions defined as follows*

$$R_n\Big(\sum_{A \subseteq \{1,\dots,n\}} x_A W_A(t)\Big) = \sum_{i=1}^{n} x_{\{i\}} W_{\{i\}}(t).$$

In the special case of X - a Hilbert space, R_n is an orthogonal projection of norm 1.
The K-convexity constant of X is

$$K(X) = \sup_n \|R_n\|.$$

In the next lecture we will prove, that if X is an m dimensional subspace of $L_1(0,1)$ then X $(1+\epsilon)$-embeds into ℓ_1^k, whenever $k \geq c(\epsilon) K^2(X) m$. In addition we will show that such an X satisfies $K(X) \leq \sqrt{\log(m)}$. Hence, X $(1+\epsilon)$-embeds into $\ell_1^{c(\epsilon)m\log(m)}$.

Exercise: Prove $K(X) = K(X^*)$. *Hint:* Show that $L_2(\Omega_n, X^*) = L_2(\Omega_n, X)^*$.

Claim 1.5. *There exist a constant $C > 0$, such that for all $1 < p < \infty$*

$$K(L_p) \leq C \max\{\sqrt{p}, \sqrt{\frac{p}{p-1}}\}.$$

Proof. Fix n and consider $f : \Omega_n \to L_p$. We already mentioned that it can be represented as

$$f(t) = \sum_{A\subseteq\{1,\dots,n\}} x_A W_A(t), \quad x_A \in L_p.$$

By the definition of R_n, if $p > 2$,

$$\begin{aligned}
\|R_n f\|_{L_2(L_p)} &= \|\sum_{i=1}^{n} x_{\{i\}} r_i(t)\|_{L_2(L_p)} = \Big(\int_{\Omega_n} (\int |\sum_{i=1}^{n} x_{\{i\}}(s) r_i(t)|^p ds)^{2/p} dt\Big)^{1/2} \leq \\
\textit{Holder's inequality} &\leq \Big(\int\int_{\Omega_n} |\sum_{i=1}^{n} x_{\{i\}}(s) r_i(t)|^p dt\, ds\Big)^{1/p} \leq \\
\textit{Khinchine's inequality} &\leq K_p \Big(\int (\sum_{i=1}^{n} x_{\{i\}}^2(s))^{p/2}\, ds\Big)^{1/p} \leq \\
&= K_p \Big(\int \Big(\int_{\Omega_n} |\sum_{i=1}^{n} r_i(t) x_{\{i\}}^2(s)|\, dt\Big)^{p/2} ds\Big)^{1/p} \leq \\
&\leq K_p \Big(\int \Big(\int_{\Omega_n} |\sum_{A\subseteq\{1,\dots,n\}} W_A(t) x_A(s)|^2\, dt\Big)^{p/2} ds\Big)^{1/p} \leq \\
\textit{triangle inequality in } L_{p/2} &\leq K_p \Big(\int_{\Omega_n} \Big(\int |\sum_{A\subseteq\{1,\dots,n\}} W_A(t) x_A(s)|^p\, ds\Big)^{2/p} dt\Big)^{1/2} = \\
&= K_p \Big(\int_{\Omega_n} \|\sum_{A\subseteq\{1,\dots,n\}} W_A(t) x_A(s)\|_{L_p}^2\, dt\Big)^{1/2}.
\end{aligned}$$

That is, for $p > 2$ we have

$$\|R_n f\|_{L_2(L_p)} \leq K_p \|f\|_{L_2(L_p)}.$$

Hence,

$$K(L_p) \leq K_p \leq C\sqrt{p}.$$

For $1 < p < 2$ we just use the duality $K(L_p) = K(L_p^*)$:

$$K(L_p) = K(L_{\frac{p}{p-1}}) \le C\sqrt{\frac{p}{p-1}}.$$

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Remark:

1. If X is a subspace of Y, then $K(X) \le K(Y)$.
2. Let $T : X \to Y$ be a one-to-one and onto, then

$$K(X) \le \|T\| \cdot \|T^{-1}\| \cdot K(Y).$$

Exercise: Let $p > 1$ and $I : \ell_1^k \to \ell_p^k$ an identity operator, then

$$\|I\| \cdot \|I^{-1}\| = k^{1-\frac{1}{p}}.$$

Combining all the results we get the estimation for $p = 1$

$$K(\ell_1^k) \le \min_{1<p<\infty} k^{1-\frac{1}{p}} K(\ell_p^k) \le c \min_{1<p<2} k^{1-\frac{1}{p}} \sqrt{\frac{p}{p-1}} \le c \cdot e \cdot \sqrt{\log(k)},$$

by substituting $p = \frac{\log(k)}{\log(k)-1}$.

Corollary 1.6. *Let X be an m-dimensional subspace of L_1, then*

$$K(X) \le C\sqrt{\log(m)}.$$

Proof. We have shown that such an X 2-embeds into $\ell_1^{cm^2}$. Hence,

$$K(X) \le 2 \cdot K(\ell_1^{cm^2}) \le C \cdot \sqrt{\log(m)}.$$

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Exercise: Show, that $K(\ell_1^k) \ge c\sqrt{\log(k)}$.

For a general Banach space X of dimension m it is known that

$$K(X) \le c \log(m),$$

which is the best possible estimation.

References

[MS] V. Milman and G. Schechtman, *Asyptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, 1200, Springer-Verlag, Berlin, 1986

[P] G. Pisier, *The volumes of convex bodies and Banach space geometry*, Cambridge University Press, Cambridge 1989

[S] G. Schechtman, *Concentration, results and applications*, available on-line at: http://www.wisdom.weizmann.ac.il/ gideon/papers/concentrationNov19.ps