SEMISIMPLE LIE GROUPS SATISFY PROPERTY RD, A SHORT PROOF

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Abstract. We give a short elementary proof of the fact that connected semisimple real Lie groups satisfy property RD. The proof is based on a process of linearization.

1. INTRODUCTION

A length function $L : G \to \mathbb{R}^+_0$ on a locally compact group $G$ is a measurable function satisfying

1. $L(e) = 0$ where $e$ is the neutral element of $G$
2. $L(g^{-1}) = L(g)$
3. $L(gh) \leq L(g) + L(h)$.

A unitary representation $\pi : G \to U(H)$ on a complex Hilbert space has property RD with respect to $L$ if there exists $C > 0$ and $d \geq 1$ such that for each pair of unit vectors $\xi$ and $\eta$ in $H$, we have

$$\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1 + L(g))^d} \, dg \leq C$$

where $dg$ is a (left) Haar measure on $G$. We say that $G$ has property RD if its regular representation has property RD with respect to $L$. First established for free groups by U. Haagerup in [6], property RD has been introduced and studied as such by P. Jolissaint in [8], who notably established it for groups of polynomial growth, and for classical hyperbolic groups. See [12] (Chap. 8, p.69), and for more details.

If $\pi$ denotes a unitary representation on a Hilbert space $H$, then $\overline{\pi}$ denotes its conjugate representation on the conjugate Hilbert space $\overline{H}$. The process of linearisation consists in working with $\sigma : G \to U(\overline{H} \otimes H)$ the unitary representation $\sigma = \overline{\pi} \otimes \pi$, see [3] (section 2.2).

A connected semisimple real Lie group with finite center can be written $G = KP$ where $K$ is a compact connected subgroup, and $P$ a closed amenable subgroup. We denote by $\Delta_P$ the right-modular function of $P$. Extend to $G$ the map $\Delta_P$ of $P$ as $\Delta : G \to \mathbb{R}^+_0$ with $\Delta(g) = \Delta(kp) := \Delta_P(p)$. It’s well defined because $K \cap P$ is compact (observe that $\Delta_P|_{K \cap P} = 1$). The quotient $G/P$ carries a unique quasi-invariant measure $\mu$, such that the Radon-Nikodym derivative at $(g, x) \in G \times G/P$ denoted by

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c(g,x) = \frac{dg \cdot \mu}{dx}(x) \text{ with } g \mu(A) = \mu(g^{-1}A), \text{ satisfies } \frac{dg \cdot \mu}{dx}(x) = \frac{\Delta(gx)}{\Delta(x)} \text{ for all } g \in G \text{ and } x \in G/P, \text{ (notice that for all } g \in G, \text{ the function } x \in G/P \mapsto \frac{\Delta(gx)}{\Delta(x)} \in \mathbb{R}_+^* \text{ is well defined)}. \text{ We refer to } [2] \text{[Appendix B, Lemma B.1.3, p. 344-345] for more details. Consider the quasi-regular representation } \lambda_{G/P} : G \to U(L^2(G/P)) \text{ associated to } P, \text{ defined by } (\lambda_{G/P}(g)\xi)(x) = c(g^{-1},x)\xi(g^{-1}x). \text{ Denote by } dk \text{ the Haar measure on } K, \text{ and under the identification } G/P = K/(K \cap P), \text{ denote by } d[k] \text{ the measure } \mu \text{ on } G/P.

The well-known Harish-Chandra function is defined by } \Xi(g) := \langle \lambda_{G/P}(g)1_{G/P}, 1_{G/P} \rangle \text{ where } 1_{G/P} \text{ denotes the characteristic function of the space } G/P.

In the rest of the paper we set } \sigma = \overline{\lambda_{G/P}} \otimes \lambda_{G/P}. \text{ Observe that } L^2(G/P) \otimes L^2(G/P) \cong L^2(G/P \times G/P), \text{ via: } \xi \otimes \eta \mapsto \left((x,y) \mapsto \overline{\xi(x)}\eta(y)\right). \text{ Notice that } \sigma \text{ preserves the cone of positive functions on } L^2(G/P \times G/P).

Let } G \text{ be a (non compact) connected semisimple real Lie group. Let } \mathfrak{g} \text{ be its Lie algebra. Let } \theta \text{ be a Cartan involution. Define the bilinear form denoted by } (X,Y) \text{ such that for all } X,Y \in \mathfrak{g}, \ (X,Y) = -B(X,\theta(Y)) \text{ where } B \text{ is the Killing form. Set } |X| = \sqrt{\langle X,X \rangle}. \text{ Write } \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} \text{ the eigenvector space decomposition associated to } \theta \text{ (l for the eigenvalue 1). Let } K \text{ be the compact subgroup defined as the connected subgroup whose Lie algebra l is the set of fixed points of } \theta. \text{ Fix } a \subset \mathfrak{p} \text{ a maximal abelian subalgebra of } \mathfrak{p}. \text{ Consider the roots system } \Sigma \text{ associated to } a \text{ and let } \Sigma^+ \text{ be the set of positive roots, and define the corresponding positive Weyl chamber as } \mathfrak{a}^+ := \{H \in a, \alpha(H) > 0, \forall \alpha \in \Sigma^+\}.

Let } A^+ = Cl(\exp(a^+)), \text{ where } Cl \text{ denotes the closure of } \exp(a^+). \text{ Consider the corresponding polar decomposition } KA^+K. \text{ Then define the length function } L(g) = L(k_1e^Hk_2) := |H| \text{ where } g = k_1e^Hk_2 \text{ with } e^H \in A^+. \text{ Notice that } L \text{ is } K \text{ bi-invariant. The disintegration of the Haar measure on } G \text{ according to the polar decomposition is } \\\ \text{ where } dk \text{ is the Haar measure on } K, \ dH \text{ the Lebesgue measure on } a^+, \text{ and } J(H) = \prod_{\alpha \in \Sigma^+} \left(\frac{e^{\alpha(H)} - e^{-\alpha(H)}}{2}\right)^{n_\alpha} \text{ where } n_\alpha \text{ denotes the dimension of the root space associated to } \alpha. \text{ See } [3] \text{[Chap.V, section 5, Proposition 5.28, p.141-142], } [5] \text{[Chap. 2, §2.2, p.65] and } [5] \text{[Chap 2, Proposition 2.4.6, p.73] for more details.}

The aim of this note is to give a short proof of the following known result (\[4], [7]).

**Theorem.** (C. Herz.) Let } G \text{ be a connected real semisimple Lie group with finite center. Then } G \text{ has property RD with respect to } L.

2. Proof

Proof. We shall prove that the quasi-regular representation has property RD with respect to $L$ defined above. This implies that the regular representation has property RD with respect to $L$ by Lemma 2.3 in [11]. Write $G = KP$ where $K$ is a compact subgroup and $P$ is a closed amenable subgroup of $G$. It’s sufficient to prove that there exists $d_0 \geq 1$ and $C_0 \geq 0$ such that $\int_G \frac{(\lambda_{G/P}(g)\xi)^2}{(1 + L(g))^{d_0}} dg < C_0$, for positive functions $\xi$, with $\|\xi\| = 1$.

Take $\xi \in L^2(G/P)$ such that $\xi \geq 0$, and $\|\xi\| = 1$. Define the function

$$F : G/P \times G/P \to \mathbb{R}_+$$

$$(x, y) \mapsto \int_K \sigma(k)(\xi \otimes \xi)(x, y) dk.$$ 

For all $(x, y) \in G/P \times G/P$, we have by the Cauchy-Schwarz inequality:

$$\int_K \sigma(k)(\xi \otimes \xi)(x, y) dk = \int_K \xi(k^{-1}x)\xi(k^{-1}y) dk \leq \left( \int_K \xi^2(k^{-1}x)dk \right)^{\frac{1}{2}} \left( \int_K \xi^2(k^{-1}y)dk \right)^{\frac{1}{2}}.$$ 

Observe that the function $f : x \in G/P \mapsto \int_K \xi^2(k^{-1}x)dk \in \mathbb{R}_+$ is constant. Indeed, fix $x \in G/P$ and let $y \in G/P$. Write $y = hx$ for some $h \in K$ (as $K$ acts transitively on $G/P$). By invariance of the Haar measure we have $f(y) = \int_K \xi^2(k^{-1}y)dk = \int_K \xi^2(k^{-1}hx)dk = \int_K \xi^2(k^{-1}x)dk = f(x)$. If $e$ is the neutral element in $G$, we write $[e] \in G/P$. We have for all $x \in G/P$, $f(x) = f([e])$.

Hence, for all $x \in G/P$ we have

$$\int_K \xi^2(k^{-1}x)dk = \int_K \xi^2(k^{-1}[e])dk = \int_{K/K\cap P} \xi^2([k^{-1}])d[k] = \|\xi\|^2 = 1.$$ 

Therefore $||F||_\infty := \sup \{F(x, y), (x, y) \in G/P \times G/P\} \leq 1$. Hence $0 \leq F \leq 1_{G/P \times G/P}$, where $1_{G/P \times G/P}$ denotes the characteristic function of $G/P \times G/P$.

Let $r$ be the number of indivisible positive roots in $a$. We know that there exists $C > 0$ such that for all $H \in a$ where $e^H \in A^+$ we have

$$\Xi(e^H) \leq Ce^{-\rho(H)} (1 + L(e^H))^r$$ 

with $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} n_{\alpha} \alpha \in a^+$, see [5][Chap 4, Theorem 4.6.4, p.161]. Hence for $d_0 > \dim(a) + 2r$, we have

$$\int_{a^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^{d_0}} J(H) dH < \infty.$$
We obtain for all $d \geq 0$ and for all positive functions $\xi$, with $||\xi|| = 1$:

$$
\int_G \frac{\langle \lambda_{G/P}(g) \xi, \xi \rangle^2}{(1 + L(g))^d} dg = \int_G \frac{\langle \lambda_{G/P}(g) \xi, \xi \rangle \langle \lambda_{G/P}(g) \xi, \xi \rangle}{(1 + L(g))^d} dg
$$

$$
= \int_G \frac{\langle \sigma(g) \xi \otimes \xi, \xi \otimes \xi \rangle}{(1 + L(g))^d} dg
$$

$$
= \int_K \int_{a^+} \int_K \frac{\langle \sigma(k_1 e^H k_2) \xi \otimes \xi, \xi \otimes \xi \rangle}{(1 + L(k_1 e^H k_2))^d} J(H) dk_1 dH dk_2
$$

$$
= \int_K \int_{a^+} \int_K \frac{\langle \sigma(e^H) \sigma(k_2) \xi \otimes \xi, \sigma(k_1^{-1}) \xi \otimes \xi \rangle}{(1 + L(e^H))^d} J(H) dk_1 dH dk_2
$$

$$
= \int_{a^+} \frac{\langle \sigma(e^H) F, F \rangle}{(1 + L(e^H))^d} J(H) dH
$$

$$
\leq \int_{a^+} \frac{\langle \sigma(e^H) 1_{G/P \times G/P}, 1_{G/P \times G/P} \rangle}{(1 + L(e^H))^d} J(H) dH
$$

$$
= \int_{a^+} \frac{\langle \lambda_{G/P}(e^H) 1_{G/P}, 1_{G/P} \rangle^2}{(1 + L(e^H))^d} J(H) dH.
$$

$$
= \int_{a^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^d} J(H) dH
$$

Take $d_0 > \text{dim}(a) + 2r$ and $C_0 = \int_{a^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^d_0} J(H) dH$. We have fund $d_0 \geq 1$ and $C_0 > 0$ such that for all positive functions $\xi$ in $L^2(G/P)$ with $||\xi|| = 1$, we have

$$
\int_G \frac{\langle \lambda_{G/P}(g) \xi, \xi \rangle^2}{(1 + L(g))^{d_0}} dg \leq C_0
$$

as required. \qed

**Remark 2.1.** The same approach applies to algebraic semisimple Lie groups over local fields. See [17] [section 1, (1.3)] and [13] [Lemme II.1.5.]

**Remark 2.2.** It’s not hard to see that this approach shows that the representations of the principal series of $G$ (of class one, see [5] (3.1.12) p.103) satisfy also property RD.

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**References**


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