BRAIDLESS WEIGHTS, MINIMAL REPRESENTATIVES AND THE WEA YL GROUP MULTIPLE DIRICHLET SERIES

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Abstract. For a semisimple Lie algebra admitting a good enumeration, we prove a parameterization for the elements in its Weyl group. As an application, we give coordinate-free comparison between the crystal graph description (when it is known) and the Lie-theoretic description of the Weyl group multiple Dirichlet series in the stable range.

1. Introduction

The purpose of this paper is to give a comparison between two descriptions of the Weyl group multiple Dirichlet series. Weyl group multiple Dirichlet series (associated to a root system Φ and a positive integer n) are Dirichlet series in r complex variables which initially converge on a cone in \( \mathbb{C}^r \), possess analytic continuation to a meromorphic function on the whole complex space, and satisfy functional equations whose action on \( \mathbb{C}^r \) is isomorphic to the Weyl group of Φ. They arise as Whittaker coefficients of Eisenstein series on covers of reductive groups and have applications in analytic number theory.

We begin by describing the shape of the Weyl group multiple Dirichlet series. Given a number field \( F \) containing the \( 2^n \)-th roots of unity and a finite set of places \( S \) of \( F \) (chosen with certain restrictions), let \( \mathcal{O}_S \) denote the ring of \( S \)-integers in \( F \) and \( \mathcal{O}_S^\times \) the units in this ring. Then to any \( r \)-tuple of nonzero \( \mathcal{O}_S \) integers \( \mathbf{m} = (m_1, \ldots, m_r) \), we associate a Weyl group multiple Dirichlet series in \( r \) complex variables \( s = (s_1, \ldots, s_r) \) of the form

\[
Z_{\Psi}(s_1, \ldots, s_r; m_1, \ldots, m_r) = Z_{\psi}(s; m) = \sum_{c = (c_1, \ldots, c_r) \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r} \frac{H^{(n)}(c; m)\Psi(c)}{|c_1|^{2s_1} \cdots |c_r|^{2s_r}},
\]

where the coefficients \( H^{(n)}(c; m) \) carry the main arithmetic content. The function \( \Psi(c) \) guarantees the numerator of our series is well-defined up to \( \mathcal{O}_S^\times \) units. Finally \( |c_i| = |c_i|_S \) denotes the norm of the integer \( c_i \) as a product of local norms on \( F_S = \prod_{\nu \in S} F_\nu \).

The coefficients \( H^{(n)}(c; m) \) are not multiplicative, but nearly so and can be reconstructed from coefficients of the form

\[
H^{(n)}(p^k; p^l) := H^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}),
\]

where \( p \) is a fixed prime in \( \mathcal{O}_S \), \( k = (k_1, \ldots, k_r), l = (l_1, \ldots, l_r), \) \( k_i = \text{ord}_p(c_i) \) and \( l_i = \text{ord}_p(m_i) \).

There are three approaches to defining these prime-power contributions so that \( Z_{\Psi}(s; m) \) admits analytic continuation to \( \mathbb{C}^r \) and satisfies the desired functional equations.

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(1) When $n$ is sufficiently large (see Section 5.2 below), the $p$-parts admit a simple Lie-theoretic definition. This is done in [BBF06, BBF08]. For a fixed $l$, the $k$’s such that $H^{(n)}(p^k; p^l) \neq 0$ have a bijection with the Weyl group $W(\Phi)$.

(2) Chinta-Gunnells [CG10] use a remarkable action of the Weyl group to define the coefficients $H^{(n)}(p^k; p^l)$ as an average over elements of the Weyl group for any root system $\Phi$ and any integer $n \geq 1$, from which functional equations and analytic continuation of the series $Z$ follows.

(3) For $\Phi$ of type $A$ and any $n \geq 1$, the prime-power coefficients are defined as a sum over basis vectors in a highest weight representation for $GL(r+1, \mathbb{C})$ associated to the fixed $r$-tuple $l$ ([BBF11b, BBF11a]). By choosing a nice decomposition of the longest element, this can also be described using a combinatorial model for highest weight representations – the Berenstein-Zelevinsky-Littelmann patterns. The same is carried out for type $C$ and odd $n$ in [FZ15] (an inductive formula is also established for even degree covers). A conjecture of this form for type $D$ is stated in [CG12].

It is widely believed that these three definitions agree. The equivalence between (1) and (2) is shown in [Fri17], generalizing the approach in [CFG08]. In type $A$, the equivalence between (2) and (3) follows from [McN11] and [CO13], by interpreting both descriptions as values of unramified Whittaker functions. The equivalence between (1) and (3) can be found in [BBF08] Section 8 for type $A$ and [BBF11] Section 4 for type $C$. In both cases, the proofs rely on explicit realization of the root systems in $\mathbb{R}^r$.

In [BBF11], the authors ask if a coordinate-free proof for the equivalence between (1) and (3) exists. This is the question we would like to address.

Assume that $n$ is sufficiently large. Our comparison is based on the following observations.

- The BZL-patterns having nontrivial contribution in the crystal graph description correspond to the orbit of the highest weight vector under the Weyl group. This provides a natural candidate for such a bijection. But this bijection is not explicit.
- In the crystal graph description, one first decorate each pattern, and the contribution is expressed in terms of $n$-th order Gauss sums according to the decoration. One easily sees that for a pattern to have nontrivial contribution, it must be a “stable pattern” (see Definition 4.2). We show that decorations for stable patterns must have much nicer shapes (see Lemma 4.5).

Our goal is to establish a bijection between the set of possible decorations and the Weyl group directly. By doing this, we can give an explicit bijection between the set of stable patterns and the Weyl group (see Proposition 4.6), without passing to the orbit of the highest weight vector. The comparison between the crystal graph description and the Lie-theoretic description of the Weyl group multiple Dirichlet series is very natural in this setting.

In the process of confirming this bijection, we discover that the nice decomposition of the longest element plays an important role. A bijection between a set of decorations and the Weyl group is also established for root systems admitting good enumerations (see Section 3.3 below), not just for root systems of type $A$ and $C$.

The rest of the paper is organized as follows. In Section 3, we describe the parameterization of the Weyl group for certain root systems as a set of decorations. This section is also of independent interest. The cases required for applications are highlighted in Section 3.4 and the rest is given in Appendix A. The study of BZL-patterns is carried out in Section 4. We recall the definition and properties of BZL-patterns. We define stable and unstable patterns,
and prove various properties. In particular, we give a natural bijection between the Weyl group and the set of stable patterns. Section 5 is devoted to Weyl group multiple Dirichlet series. We first review the Lie-theoretic description in the stable range and the known cases of the crystal graph description. Our main result, i.e. the comparison between these two descriptions in the stable range, is done in Section 5.4. The comparison can be done term by term canonically.

Some of the results in this paper are only stated for type $A$ and $C$. We try to minimize the use of case-by-case check in the proofs as some of them actually work in a more general setting. Our results also suggest what the crystal graph description should look like for general root systems, at least for stable patterns.

2. Notations and Preliminaries

In this section, we set up notations for use in the sequel.

2.1. Root system. Let $\Phi$ be a reduced root system, $\Phi^+$ and $\Phi^-$ a choice of positive and negative roots respectively, and $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ the set of simple roots. We denote by $\Phi^\vee$ the coroots and $\alpha \mapsto \alpha^\vee$ the bijection between $\Phi$ and $\Phi^\vee$. We use $\langle \cdot, \cdot \rangle : \Phi \times \Phi^\vee \to \mathbb{Z}$ to denote the canonical pairing between $\Phi$ and $\Phi^\vee$.

Let $\omega_1, \cdots, \omega_r$ be the fundamental dominant weights, which satisfy $\langle \omega_i, \alpha^\vee_j \rangle = \delta_{ij}$ ($\delta_{ij} =$ Kronecker delta).

Let $\Lambda$ be the weight lattice, generated by the $\omega_i$. Let

\[ \rho = \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \]

Given a weight $\lambda$, define $d_\lambda(\alpha^\vee) = \langle \lambda + \rho, \alpha^\vee \rangle$.

Let $W = W(\Phi)$ be the Weyl group of the root system $\Phi$. It is generated by the simple reflections $s_{\alpha_i} = s_i$ for $i = 1, \cdots, r$. (We also use $s_i$ as complex variables later. But this will not cause any confusion.) The action of $s_i$ on $\Lambda$ is given by $\lambda \mapsto s_i(\lambda) := \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$.

Let $\ell(\cdot)$ denote the length function on $W$. Let $w_0$ denote the element of the longest length in $W$ and denote that length by $N$. For $w \in W$, define $\Phi(w) = \{ \alpha \in \Phi^+: w(\alpha) \in \Phi^- \}$.

We also need the following results.

Lemma 2.1 ([Bum13] Proposition 20.10). If $w = s_{i_1} \cdots s_{i_N}$ is a reduced decomposition of $w$, then

\[ \Phi(w) = \{ \alpha_{i_N}, s_{i_N}(\alpha_{i_{N-1}}), \cdots, s_{i_N} s_{i_{N-1}} \cdots s_{i_2} \alpha_{i_1} \}. \]

Lemma 2.2. For any $\alpha, \beta \in \Phi$,

\[ (s_\alpha(\beta))^\vee = s_{\alpha^\vee}(\beta^\vee). \]

Proof. This is proved in the last two lines of [Spr79] Page 4. \qed

When $\alpha = \alpha_i$ is a simple root, we write $s_i(\beta^\vee) = s_{\alpha_i^\vee}(\beta^\vee)$. The meaning of $s_i$ (either $s_{\alpha_i}$ or $s_{\alpha_i^\vee}$) is always clear in the context.
2.2. Hilbert Symbols. Let $n > 1$ be an integer and let $F$ be a number field containing the $n$-th roots of unity. Let $S$ be a finite set of places of $F$ such that $S$ contains all Archimedean places, all places ramified over $\mathbb{Q}$, and is sufficiently large that the ring of $S$-integers $\mathcal{O}_S$ is a principal ideal domain. Embed $\mathcal{O}_S$ in $F_S = \prod_{\nu \in S} F_{\nu}$ diagonally.

The product of local Hilbert symbols gives rise to a pairing

$$(a, b)_S : F_S^\times \times F_S^\times \to \mu_n, \quad (a, b)_S = \prod_{\nu \in S} (a, b)_\nu.$$

A subgroup $\Omega$ of $F_S^\times$ is called isotropic if $(a, b)_S = 1$ for all $a, b \in \Omega$.

2.3. Gauss sum. If $a \in \mathcal{O}_S$ and $b$ is an ideal of $\mathcal{O}_S$, let $(a/b)$ be the $n$th order power residue symbol as defined in [BB06]. (This depends on $S$, but we suppress this dependence from the notation.) Let $\psi$ be a nontrivial additive character $\psi$ of $F_S$ such that $\psi(xO_S) = 1$ if and only if $x \in O_S$. If $a, c \in \mathcal{O}_S$ and $c \neq 0$, and if $t$ is a positive integer, the Gauss sum is defined by

$$g_t(a,c) = \sum_{d \mod c} \left( \frac{d}{cO_S} \right)^t \psi \left( \frac{ad}{c} \right).$$

3. Braidless Weights and Minimal Representatives

In this section, we describe a parametrization of elements in the Weyl group for certain root systems. The only cases that appear in the application are given in Section 3.4. The reader can safely skip the materials before Section 3.4.

3.1. Braidless weights. Recall that a fundamental weight $\omega$ is braidless if for any $\tau \in W$ the following holds ([Lit98] Section 3):

if $\alpha, \gamma \in \Delta$ are such that $\langle \tau(\omega), \alpha^\vee \rangle > 0, \langle \tau(\omega), \gamma^\vee \rangle > 0$, then $\langle \gamma, \alpha^\vee \rangle = 0$.

We call a simple root $\alpha$ braidless if the corresponding fundamental weight $\omega_\alpha$ is so.

The set of braidless weights is determined by [Lit98] Lemma 3.1. Here we use the enumeration of fundamental weights as in [Bou02].

**Lemma 3.1** ([Lit98] Lemma 3.1). A fundamental weight $\omega$ of a simple Lie algebra $\mathfrak{g}$ is braidless if and only if $\omega$ is a minuscule weight of $\mathfrak{g}$, or $\omega = \omega_1$ for $\mathfrak{g}$ of type $B_n$, or $\omega = \omega_n$ for $\mathfrak{g}$ of type $C_n$, or $\omega$ is arbitrary and $\mathfrak{g}$ is of rank at most two.

It is not hard to find all braidless weights (see also [BF15] Section 5.2). A fundamental weight $\omega$ is braidless if and only if it is in the following collection:

- $\Phi$ is of type $A$,
- $\Phi$ is of type $B$ or $C$ and $\omega = \omega_1$ or $\omega_r$,
- $\Phi$ is of type $D$ and $\omega = \omega_1, \omega_r-1$ or $\omega_r$,
- $\Phi = E_6$ and $\omega = \omega_1$ or $\omega_6$, or $\Phi = E_7$ and $\omega = \omega_7$,
- $\Phi = G_2$.

Suppose $\omega = \omega_1$ is braidless. Let $W_\omega$ be the subgroup of $W$ generated by $\{s_\alpha : \alpha \in \Delta - \{\alpha_1\} \}$. Denote by $^{\omega}W$ the minimal representatives of the left $W_\omega$-classes in $W$. By [Lit98] Lemma 3.2, if $\tau \in ^{\omega}W$, then a reduced decomposition of $\tau$ is unique up to the exchange of orthogonal simple reflections.
3.2. Parameterizations. We describe a canonical parametrization for elements in $\omega W$ if $\omega$ is braidless. Our main result is Theorem 3.2. We also give another parametrization in terms of graphs (Corollary 3.7). This is motivated by the Weyl group multiple Dirichlet series. (See [CG12] Section 3; also compare their description and Section A.3 below.)

Let $\tau_\ell \in \omega W$ be the longest element and fix a reduced decomposition $\tau_\ell = s_{i_1} \cdots s_{i_N}$.

Notice that any series of exchange of orthogonal simple reflections induces a permutation of $\{1, \ldots, N\}$. Any other reduced decomposition of $\tau_\ell$ is of the form $s_{i_{\sigma(1)}} \cdots s_{i_{\sigma(N)}}$ for a unique permutation $\sigma$. Let $S_\omega$ be the set of such permutations.

Notice that $\sigma(1) = 1$ for all $\sigma \in S_\omega$. Another observation is that if $j < k$ and $s_{i_j}$ and $s_{i_k}$ do not commute, then $\sigma^{-1}(j) < \sigma^{-1}(k)$ for any $\sigma \in S_\omega$. (Note that $\sigma^{-1}(j)$ indicates the position of $s_{i_j}$ in new reduced expression $s_{i_{\sigma(1)}} \cdots s_{i_{\sigma(N)}}$. And $\sigma(j) < \sigma(k)$ may not be true for some $\sigma \in S_\omega$.)

Define $C_\omega$ to be a set of certain subsets of $\{1, \ldots, N\}$ as follows. The set $C \in C_\omega$ if and only if it is of the form $\{\sigma(1), \ldots, \sigma(k)\}$ for some $k \geq 0$ and $\sigma \in S_\omega$. (If $k = 0$, we set $C = \emptyset$.) Define a map

$$f_\omega : C_\omega \to W, \quad C = \{\sigma(1), \ldots, \sigma(k)\} \mapsto \tau_C = s_{i_{\sigma(1)}} \cdots s_{i_{\sigma(k)}}.$$ 

Clearly, this map does not depend on the choice of $\sigma$. If we start with another reduced decomposition of $\tau_\ell$, then the map $f_\omega$ changes by a conjugation induced by some $\sigma \in S_\omega$.

**Theorem 3.2.**

1. For $C \in C_\omega$, we have $\tau_C \in \omega W$, and $s_{i_{\sigma(1)}} \cdots s_{i_{\sigma(k)}}$ is a reduced expression.
2. The map $f_\omega$ is a bijection.

**Example 3.3.** Let $g = so_8$ and $\omega = \omega_4$ (see Section A.3 for the enumeration). The longest element in $\omega W$ is $\tau_\ell = s_{43}s_1s_2s_3s_4$. Then

$$C_\omega = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5, 6\},$$

and

$$\omega W = \{I, s_4, s_{43}s_1, s_{43}s_2, s_{43}s_1s_2s_3, s_{43}s_1s_2s_3s_4\}.$$ 

**Proof.** (1) Clearly $s_{i_{\sigma(1)}} \cdots s_{i_{\sigma(k)}}$ is a reduced expression. Otherwise, $s_{i_{\sigma(1)}} \cdots s_{i_{N(k)}}$ is not reduced. To show that $\tau_C$ is a minimal representative, we need to show that $\Phi(\tau_C^{-1}) \cap (\Delta - \{\alpha_i\}) = \emptyset$ ([Cas] Lemma 1.1.2). Without loss of generality, we may assume $\sigma$ is the identity element. By Lemma 2.1,

$$\Phi(\tau_\ell^{-1}) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})\}$$

and

$$\Phi(\tau_C^{-1}) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}.$$ 

Thus $\Phi(\tau_C^{-1}) \subset \Phi(\tau_\ell^{-1})$. The fact that $\tau_\ell$ is minimal implies that $\tau_C$ is minimal as well.

(2) We first show that this map is a surjection. Let $\tau \in \omega W$. Let $\tau_\ell$ be the longest element in $W_\omega$. Then $w_0 = \tau_\ell \tau_\ell$ is the longest element in $W_\omega$, and $\ell(w_0) - \ell(\tau_\ell^{-1}) = \ell(w_0 \cdot \tau_\ell^{-1}).$ Thus

$$\ell(w_0) - \ell(\tau_\ell^{-1}) = \ell(\tau_\ell \tau_\ell^{-1}).$$ 

By [Cas] Lemma 1.1.2, we know $\ell(\tau_\ell \tau_\ell^{-1}) = \ell(\tau_\ell) + \ell(\tau_\ell^{-1})$ and $\ell(w_0) = \ell(\tau_\ell) + \ell(\tau_\ell^{-1}).$ Thus

$$\ell(\tau_\ell) + \ell(\tau_\ell^{-1}) = \ell(\tau_\ell).$$
This means that if we choose reduced decompositions \( s_{j_1} \cdots s_{j_k} \) and \( s_{j_{k+1}} \cdots s_{N} \) for \( \tau \) and \( \tau^{-1} \tau_\ell \) respectively, then \( s_{j_1} \cdots s_{j_{k}} \) is a reduced decomposition for \( \tau_\ell \). Thus there is a permutation \( \sigma \in S_\omega \) such that \( j_1 = \sigma(i) \) and \( \tau = s_{i_\sigma(1)} \cdots s_{i_\sigma(k)} \).

Next we show \( f_\omega \) is injective. Without loss of generality, we show that if there are two sets
\[
\{1, \cdots, k\}, \quad \{\sigma(1), \cdots, \sigma(k)\}
\]
such that \( s_{i_1} \cdots s_{i_k} = s_{i_\sigma(1)} \cdots s_{i_\sigma(k)} \), then \( \{1, \cdots, k\} = \{\sigma(1), \cdots, \sigma(k)\} \). We argue by induction on \( k \). If \( k = 0 \) or \( 1 \), the result is clear. Now assume the result is true for \( k - 1 \). From \( s_{i_1} \cdots s_{i_k} = s_{i_\sigma(1)} \cdots s_{i_\sigma(k)} \), we know the multi-sets \( \{i_1, \cdots, i_k\} \) and \( \{i_\sigma(1), \cdots, i_\sigma(k)\} \) are the same. Let \( j \) be the maximal index in \( \{1, \cdots, k\} \) such that \( \sigma(i) = i_\sigma(k) \). Then \( s_{ij} \) commutes with \( s_{i_{j+1}}, \cdots, s_{i_k} \) and thus \( s_{i_1} \cdots s_{i_{j-1}} s_{ij} s_{i_{j+1}} \cdots s_{i_k} = s_{i_\sigma(1)} \cdots s_{i_\sigma(k-1)} \). Here \( \hat{s}_{ij} \) means that the corresponding term is omitted. By induction we know \( \{1, \cdots, j, \cdots, k\} = \{\sigma(1), \cdots, \sigma(k - 1)\} \).

It remains to show \( j = \sigma(k) \). If \( j \neq \sigma(k) \), then \( \sigma^{-1}(j) > k \). There are two cases to consider:

(i) If \( \sigma(k) = j + 1 \), then from \( i_j = i_\sigma(k) \), we know \( s_{ij} = s_{i_{j+1}} \). This is impossible since \( s_{i_1} \cdots s_{i_N} \) is reduced.

(ii) Suppose \( \sigma(k) > j + 1 \). Since \( s_{i_1} \cdots s_{i_N} \) is reduced, there is some \( j < j' < \sigma(k) \) such that \( s_{ij} \) and \( s_{i_{j'}} \) do not commute. We deduce that \( \sigma^{-1}(j) < \sigma^{-1}(j') < \sigma^{-1}(\sigma(k)) = k \). This is a contradiction.

\[\square\]

We now introduce another description which is slightly easier to work with. Our motivation comes from the decoration graph in [CG12] Section 3. We now explain it carefully. We again start with a reduced decomposition \( \tau_\ell = s_{i_1} \cdots s_{i_N} \) and define a directed graph \( T_{i_1, \cdots, i_N} \) (with labelling) as follows. The vertices of \( T_{i_1, \cdots, i_N} \) are \( \{1, \cdots, N\} \). An edge \( j \rightarrow k \) is in \( T_{i_1, \cdots, i_N} \) if and only if \( s_{ij} \) and \( s_{ik} \) do not commute and there exists \( \sigma \in S_\omega \) and \( 1 \leq l \leq N \) such that \( \sigma(l) = j \) and \( \sigma(l + 1) = k \). (In other words, \( s_{ij} \) and \( s_{ik} \) can be moved by a series of orthogonal permutations so that they are next to each other.) We also label the vertex \( j \) by \( s_{ij} \). It is easy to see that the graph \( T_{i_1, \cdots, i_N} \) is connected. Also notice that if \( (j \rightarrow k) \in T_{i_1, \cdots, i_N} \), then \( \sigma^{-1}(j) < \sigma^{-1}(k) \) for all \( \sigma \in S_\omega \).

**Example 3.4.** Let \( g = \mathfrak{so}_8 \) and \( \omega = \omega_4 \). The longest element is \( \tau_\ell = s_4 s_3 s_1 s_2 s_3 s_4 \). Then the graph \( T_{4,3,1,2,3,4} \) is

\[
1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow s_1 \rightarrow s_3 \rightarrow s_2 \rightarrow s_4 .
\]

Let \( \mathcal{G}_\omega \) be set of subgraphs of \( T_{i_1, \cdots, i_N} \) with the following properties:

(1) If \( T \) is not the empty subgraph of \( T_{i_1, \cdots, i_N} \), then \( 1 \in T \).

(2) If \( k \in T \), and the edge \( (j \rightarrow k) \in T_{i_1, \cdots, i_N} \), then \( j \in T \).

**Example 3.5 (Continuation of Example 3.4).** The set \( \mathcal{G}_\omega \) consists of the following subgraphs:

\[
\emptyset, \quad s_4, \quad s_4 \rightarrow s_3, \quad s_4 \rightarrow s_3, \quad s_1 \rightarrow s_4 .
\]
Lemma 3.6. There is a natural bijection between $G_\omega$ and $C_\omega$.

Proof. The map from $G_\omega$ to $C_\omega$ is clear. Let $T \in G_\omega$ and $\{j_1, \cdots, j_k\}$ be the vertices of $T$. We simply send a graph $T$ to its set of vertices. To verify that this set satisfies the desired properties, we argue by induction on $k$. If $k = 0$ or 1, this is clear. Now suppose $k \geq 2$. Recall that if $(j \to j') \in T$, then $j < j'$. Thus there must be a maximal element (say, $j_k$) in $T$ (i.e. there is no edge coming out from this maximal element). Delete $j_k$ and we obtain a graph in $G_\omega$ with shorter length. Now apply induction, and (possibly) rearrange the indices in $\{j_1, \cdots, j_{k-1}\}$, we see that there is a reduced decomposition of $\tau_\ell$ which is of the form

$$(s_{i_1} \cdots s_{i_{j_k-1}}) \cdots s_{i_k} \cdots$$

We need to show that $s_{i_{j_k}}$ can be moved (by a series of orthogonal permutations) to the position which is right after $s_{i_{j_k-1}}$. If this is not true, then there is a simple reflection $s_{i_{j'}}$ between $s_{i_{j_k-1}}$ and $s_{i_{j_k}}$ such that $s_{i_{j'}}$ does not commute with $s_{i_{j_k}}$. Choose the maximal $j'$. Then $(j' \to j_k) \in T_{i_1, \cdots, i_N}$ and thus $j' \in T$. This is a contradiction.

Conversely, given a nonempty subset $\{\sigma(1), \cdots, \sigma(k)\}$ for some $k$ and $\sigma \in S_\omega$, we can define a subgraph $T$ by taking all the possible edges between these vertices. First of all, $\sigma(1) = 1 \in T$. If there is $(j' \to \sigma(j)) \in T_{i_1, \cdots, i_N}$ for some $j'$ (might not be in $T$) and $1 \leq j \leq k$. Then $\sigma^{-1}(j') < j$ and therefore $j' = \sigma(\sigma^{-1}(j')) \in T$. This implies that $T \in G_\omega$ and we are done.

We have the following immediate corollary.

Corollary 3.7. There is a natural bijection between $\omega W$ and $G_\omega$.

Remark 3.8. If we start with a different reduced decomposition of $\tau_\ell$, then the graph is obtained by the induced action from a certain $\sigma \in S_\omega$. In Appendix A, we work out all the examples. By abuse of notations, we simply call it $T_{\tau_\ell}$. We only give the labelling without giving the indices.

3.3. Good enumeration. We now recall the notation of good enumeration in the sense of [Lit98] Section 4. Let $D$ be the Dynkin diagram of $\mathfrak{g}$. We write $D - \{\alpha\}$ for the diagram obtained by removing the node of $\alpha$ from $D$.

Suppose that the set of simple roots of $G$ admits an enumeration

$$\Delta = \{\alpha_1, \cdots, \alpha_n\}$$

such that $\alpha_i$ is braidless for $D - \{\alpha_1, \cdots, \alpha_{i-1}\}$

for all $i = 1, \cdots, n$. We call this a good enumeration.

Note that all simple Lie algebras admit such a good enumeration except the ones of type $F_4$ and $E_8$. For such good enumeration of the respective types, see [Lit98] Section 5-8. Thus for a simple Lie algebra admitting good enumerations, we obtain a natural parametrization of the Weyl group $W(\mathfrak{g})$. The reduced decomposition of the longest element $w_0$ obtained in this way is called nice decomposition.
3.4. **Examples.** To avoid interrupting the application, we only give two examples (type A and type C) which are required in Section 5. The rest is given in Appendix A.

We choose the following good enumerations:

\[
\text{Type } A_r : \quad \alpha_1 - \alpha_2 - \alpha_3 - \cdots - \alpha_r, \\
\text{Type } C_r : \quad \alpha_1 = \alpha_2 - \alpha_3 - \cdots - \alpha_r.
\] (3)

Notice that in both cases, the reduced decomposition for the longest element \( \tau_\ell \) in \( \omega W \) is unique when \( \omega = \omega_r \). Thus the corresponding parameterization for \( \omega W \) for these two types is indeed simpler.

In type A, given \( 0 \leq a_i \leq i \), define

\[
\pi_{a_i} = \begin{cases} 
  s_i \cdots s_{i-a_i+1}, & \text{if } 1 \leq a_i \leq i, \\
  e \text{ (the identity element)}, & \text{if } a_i = 0.
\end{cases}
\]

In type C, given \( 0 \leq a_i \leq 2i - 1 \), define

\[
\pi_{a_i} = \begin{cases} 
  s_i \cdots s_{i-a_i+1}, & \text{if } 1 \leq a_i \leq i, \\
  s_i \cdots s_2 s_1 s_2 \cdots s_{a_i-i+1}, & \text{if } i + 1 \leq a_i \leq 2i - 1, \\
  e \text{ (the identity element)}, & \text{if } a_i = 0.
\end{cases}
\]

Define

\[
D_r = \begin{cases} 
  \text{Type } A : \quad \{(a_1, \ldots, a_r) : 0 \leq a_i \leq i, \text{ for all } i\}, \\
  \text{Type } C : \quad \{(a_1, \ldots, a_r) : 0 \leq a_i \leq 2i - 1, \text{ for all } i\}.
\end{cases}
\]

Together with the discussion in the previous section and the result in Sections A.1 and A.2, we obtain the following consequence.

**Proposition 3.9.** The nice decompositions obtained for the enumerations in Eq. (3) are

- **Type A:** \( w_0 = (s_1)(s_2 s_1) \cdots (s_r \cdots s_1) \),
- **Type C:** \( w_0 = (s_1)(s_2 s_1 s_2) \cdots (s_r s_{r-1} \cdots s_1 \cdots s_{r-1}s_r) \).

The map

\[
D_r \to W, \quad (a_1, \ldots, a_r) \mapsto \pi_{a_1} \cdots \pi_{a_r}
\]

is a bijection.

For type A, see also [Caiar].

4. **Berenstein-Zelevinsky-Littelmann Patterns and Crystal Graphs**

4.1. **BZL patterns.** Given a semisimple algebraic group \( G \) of rank \( r \) and a simple \( G \)-module \( V_\lambda \) of highest weight \( \lambda \), we may associate a crystal graph \( B_\lambda \) to \( V_\lambda \). That is, there exists a corresponding simple module for the quantum group \( U_q(\text{Lie}(G)) \) having the associated crystal graph structure. The crystal graph encodes data from the representation \( V_\lambda \), and can be regarded as a kind of "enhanced character" for the representation (see [HK02]).

The vertices of \( B_\lambda \) are in bijection with a basis of weight vectors for the highest weight representation \( V_\lambda \). Given a vertex \( v \in B_\lambda \), let \( wt(v) \) be the weight of the corresponding weight vector. Given a simple root \( \alpha_i \), there is "colored" edge of \( B_\lambda \). Two vertices \( v_1, v_2 \) are connected by a (directed) edge from \( v_1 \) to \( v_2 \) of color \( i \) if the Kashiwara raising operator
$e_i := e_{\alpha_i}$ takes $v_1$ to $v_2$. In this case, $\text{wt}(v_2) = \text{wt}(v_1) + \alpha_i$. If the vertex $v$ has no outgoing edge of color $i$, we set $e_i(v) = 0$. The Kashiwara lowering operator is denoted by $f_i$.

Remark 4.1. Notice that we follow the formulation in [Lit98]. This would make the comparison in Section 5 more natural. The formulation in [BBF11a] takes the opposite direction. However, as explained in [BBF11a] Proposition 1, they can be tied by the use of the Schützenberger involution.

Littelmann [Lit98] gives a combinatorial model for the crystal graph as follows. Fix a reduced decomposition of the longest element $w_0$ of the Weyl group of $G$ into simple reflections:

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}.$$  

Given an element $v$ (i.e. vertex) of the crystal $B_\lambda$, let $b_1$ be the maximal integer such that $v_1 := e_{i_1}^{b_1}(v) \neq 0$. Similarly, let $b_2$ be the maximal integer such that $v_2 := e_{i_2}^{b_2}(v_1) \neq 0$. Continuing in this fashion in order of the simple reflections appearing in $w_0$, we obtain a string of non-negative integers

$$\text{BZL}(v) := (b_1, \ldots, b_N).$$

This is the so-called Berenstein-Zelevinsky-Littelmann pattern (or BZL-pattern for short). By well-known properties of the crystal graph, we are guaranteed that for any reduced decomposition of the longest element and an arbitrary element $v$ of the crystal, the path $e_{i_N}^{b_N} \cdots e_{i_1}^{b_1}(v)$ through the crystal always terminates at $v_\lambda$, corresponding to the highest weight vector of the crystal graph $B_\lambda$.

Littelmann proves that, for any fixed reduced decomposition, the set of all sequences $(b_1, \ldots, b_N)$ as we vary over all vertices of all highest weight crystals $V_\lambda$ associated to $G$ fill out the integer lattice points of a cone in $\mathbb{R}^N$. The inequalities describing the boundary of this cone depend on the choice of reduced decomposition. In general, it is a difficult task to give a set of inequalities. For the nice decomposition coming from a good enumeration (see Section 3.3), Littelmann shows that the cone is defined by a rather simple set of inequalities. We describe this for type $A$ and $C$ below, and refer this as the set of cone inequalities.

4.2. Inequalities. We now describe these two sets of inequalities. We start with the set of polytope inequalities, which depends on the highest weight $\lambda$. It is given by

$$\psi_N : \quad b_N \leq \langle \lambda, \alpha_{i_N}^\vee \rangle,$$

$$\psi_{N-1} : \quad b_{N-1} \leq \langle \lambda - b_N \alpha_{i_N}, \alpha_{i_{N-1}}^\vee \rangle,$$

$$\vdots$$

$$\psi_1 : \quad b_1 \leq \langle \lambda - b_N \alpha_{i_N} - \cdots - b_2 \alpha_{i_2}, \alpha_{i_1}^\vee \rangle.$$

The set of cone inequalities is more involved. In [Lit98], an algorithm is described to give such a set of defining inequalities. However, this set is usually far from minimal. When the root system admits a good enumeration, such a set can be found in [Lit98]. We only describe it in types $A$ and $C$ for the nice decompositions given in Proposition 3.9. The interested reader is referred to [Lit98] Section 5-8 for further details.

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We arrange a BZL pattern $BZL(v) = (b_1, b_2, \cdots, b_N)$ into an array as follows. In type $A$, we set

$$BZL(v) = \begin{pmatrix} \cdots & \cdots & \cdots \\ b_2 & b_3 & b_1 \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots \\ c_{r-1,r-1} & c_{r-1,r} & c_{r,r} \end{pmatrix}.$$ 

In type $C$, we set

$$BZL(v) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ b_2 & b_3 & b_4 & \cdots & \cdots \\ b_1 \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{r-1,r-1} & c_{r-1,r} & \bar{c}_{r-1,r-1} & \cdots & \cdots \\ c_{r,r} \end{pmatrix}.$$ 

In the following, we may use either form, depending on the context.

Note that the corresponding simple reflections used to define $b_i$ are arranged as

- Type $A$: $\begin{pmatrix} \cdots & \cdots & \cdots \\ s_2 & s_1 \end{pmatrix}$,
- Type $C$: $\begin{pmatrix} \cdots & \cdots & \cdots \\ s_2 & s_1 & s_2 \end{pmatrix}$

We now describe the set of cone inequalities. In both cases, we require entries in each row are non-negative and weakly decreasing. In other words, for $1 \leq i \leq r$,

- Type $A$: $c_{i,i} \geq c_{i,i+1} \geq \cdots \geq c_{i,r} \geq 0$,
- Type $C$: $c_{i,i} \geq c_{i,i+1} \geq \cdots \geq c_{i,r} \geq \bar{c}_{i,r-1} \geq \cdots \geq \bar{c}_{i,i} \geq 0$.

Notice that each entry $b_i$ appears on the left-hand side of a unique inequality $\phi_i$. (This does not hold in general.)

4.3. **Decorations.** In [BBF11a] page 1088, a decoration rule is introduced for type $A$ patterns in the description of the Weyl group multiple Dirichlet series. A similar decoration is also given for type $C$ patterns in [BBF12] Section 2.4. We recall it here. Such a decoration for other types necessarily requires a modification.

**Definition 4.2.** An entry $b_i$ is circled (resp. boxed) if the inequality $\phi_i$ (resp. $\psi_i$) is an equality.

4.4. **Stable and Unstable Patterns.** We now define stable and unstable patterns. The meaning of these two terms would become evident in Section 5.4.

**Definition 4.3.** We call a pattern stable if every entry has one and only one decoration. Otherwise, we call it an unstable pattern.

In particular, an zero entry is always circled. It is also not boxed in a stable pattern. It is easy to see that, for a highest weight representation $V_\lambda$ to have a stable pattern, we shall assume $\langle \lambda, \alpha_0^\vee \rangle > 0$ for all $i$. We make this assumption from now on.

We have the following observation. This result is used in Section 5.4.

**Lemma 4.4.** For any $v \in B_\lambda$ and $BZL(v) = (b_1, \cdots, b_N)$,

$$\max_j b_j \leq \langle \lambda, \alpha_0^\vee \rangle.$$ 

Here $\alpha_0^\vee$ be the highest coroot. If $b_j = \langle \lambda, \alpha_0^\vee \rangle$, then $b_j$ is boxed.
Proof. By the definition of $b_i$, it is of the form $\langle \mu, \alpha^\vee \rangle$ where $\mu$ is a weight for the highest weight representation $V_{\lambda}$ and $\alpha$ is a simple root. Thus it is sufficient to maximize $\langle \mu, \alpha^\vee \rangle$ where $\mu$ is a weight of the representation $V_{\lambda}$ and $\alpha \in \Phi$.

Without loss of generality, we may assume that $\mu$ is in the dominant Weyl chamber. Write $\mu = m_1 w_1 + \cdots + m_r w_r$ for some nonnegative integers $m_1, \cdots, m_r$. Also write $\alpha^\vee = m'_1 \alpha^\vee_1 + \cdots + m'_r \alpha^\vee_r$.

$$\langle \mu, \alpha^\vee \rangle = m'_1 m'_1 + \cdots + m'_r m'_r.$$ 

Thus we may also assume that $\alpha^\vee$ is a positive coroot.

Recall that from the theory of highest weight representation, for any positive coroots $\alpha^\vee$, $\alpha^\vee_0 - \alpha^\vee$ is a nonnegative linear combination of $\alpha^\vee_1, \cdots, \alpha^\vee_r$. Thus we may assume that the maximum is achieved when $\alpha^\vee$ is the highest coroot. Similarly, we may assume $\mu$ is the highest weight $\lambda$.

Now suppose $b_j = \langle \lambda, \alpha^\vee_0 \rangle$. We show that this entry is boxed. In fact, the polytope inequality $\psi_j$ reads

$$b_j \leq \langle \lambda - \sum_{k=j+1}^N b_k \alpha_k, \alpha^\vee_j \rangle.$$ 

Note that $\lambda - \sum_{k=j+1}^N b_k \alpha_k$ is in the weight diagram of $V_{\lambda}$. Thus

$$b_j \leq \langle \lambda - \sum_{k=j+1}^N b_k \alpha_k, \alpha^\vee_j \rangle \leq \langle \lambda, \alpha^\vee_0 \rangle = b_j.$$ 

This implies that $b_j$ is boxed.

\[\square\]

Notice that it is still possible to obtain the maximum with the non-highest weight and a non-largest coroot.

Lemma 4.5. If $b_{j-1}$ and $b_j$ are in the same row, $b_{j-1}$ is circled and $b_j$ is boxed, then the pattern is unstable.

Proof. Let $j'$ be the largest index such that $j' < j$ and $\alpha_{i_{j'}} = \alpha_{i_j}$. We explain why such a $j'$ exists. Suppose we are considering the $m$-th row. Notice that $b_j$ cannot the first entry in the $m$-th row. If $i_j = m$, then we can (at least) choose the first entry in this row. If $i_j < m$, then $a_{i_j}$ must appear again in the rows below the $m$-th row.

As $b_{j-1}$ is circled and $b_j$ is boxed,

$$b_{j-1} = b_j = \langle \lambda - \sum_{k=j+1}^N b_k \alpha_k, \alpha^\vee_j \rangle.$$ 

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On the other hand,

\[ b_{j'} \leq \langle \lambda - \sum_{k=j'+1}^{N} b_k \alpha_{i_k}, \alpha_{i_{j'}}^\vee \rangle \]

\[ = \langle \lambda - \sum_{k=j+1}^{N} b_k \alpha_{i_k}, \alpha_{i_j}^\vee \rangle - \langle \sum_{k=j'+1}^{j-2} b_k \alpha_{i_k}, \alpha_{i_{j'}}^\vee \rangle \]

\[ = b_j - 2b_j + b_{j-1} - \langle \sum_{k=j'+1}^{j-2} b_k \alpha_{i_k}, \alpha_{i_{j'}}^\vee \rangle \]

\[ = -\langle \sum_{k=j'+1}^{j-2} b_k \alpha_{i_k}, \alpha_{i_{j'}}^\vee \rangle. \]  

(4)

We now observe that in the decomposition

\[ w_0 = s_{i_1} \cdots s_{i_N}, \]

the simple reflection \( s_{i_j} \) does not appear again between \( s_{i_j} \) and \( s_{i_{j'}} \). In other words, there does not exist \( j' < k < j \) so that \( s_{i_k} = s_{i_j} \). In type \( A \) and \( C \), there are at most two \( k \) such that \( j' < k < j \) and \( \langle \alpha_{i_k}, \alpha_{i_j}^\vee \rangle \neq 0 \) – one of them is \( j - 1 \) and we assume the other one is \( k' \).

If this is the case, then \( b_{j'} \) and \( b_{k'} \) are in the same row and \( \langle \alpha_{i_{k'}}, \alpha_{i_j}^\vee \rangle = -1 \). (This can be verified for type \( A \) and \( C \) directly. However, it does not generalize to other types naively.)

If the sum in the last line of Eq. (4) is empty, then we conclude that \( b_{j'} = 0 \) and it is boxed. If the sum is not empty, then as we discuss above, only the term \( b_{k'} \langle \alpha_{i_{k'}}, \alpha_{i_j}^\vee \rangle = -b_{k'} \) is nonzero. Then we conclude that \( b_{j'} \leq b_{k'} \). Since \( b_{j'} \) and \( b_{k'} \) are in the same row, \( b_{j'} \) is boxed as well (if such a pattern exists). In either case, this is an unstable pattern.

Thus, in each row of a stable pattern, the decoration is of the following form

\[ \square \square \cdots \square \bigcirc \cdots \bigcirc \]

where the circled entries are zero.

Given a stable pattern \( v \). Define \( \text{sign}(v) = (e_1, \cdots, e_N) \in \{0,1\}^N \) where \( e_i = 1 \) if \( b_i \) is boxed. Let \( St_\lambda \) be the set of stable patterns. Define a map \( St_\lambda \to W \) by

\[ v \mapsto \text{sign}(v) = (e_1, \cdots, e_N) \mapsto w_v := s_{i_1}^{e_1} s_{i_2}^{e_2} \cdots s_{i_N}^{e_N}. \]

We also write

\[ w_v = s_{i_1} \cdots s_{i_\ell} \]  

(5)

to keep track of the simple reflections that actually appear in \( w_v \). By Proposition 3.9, this map is injective.

It is expected that the set of stable patterns corresponds to the orbit \( W v_\lambda \) of a highest weight vector. Thus we consider another natural bijection

\[ W \to W v_\lambda, \quad w \mapsto w v_\lambda. \]

We have \( \text{wt}(w v_\lambda) = w \lambda \).
To summarize, we have the following maps:

\[ \text{St}_\lambda \rightarrow W \leftarrow Wv_\lambda \rightarrow \Lambda, \]

where \( \Lambda \) is the weight lattice.

We would like to address the following two questions.

- Is the map \( \text{St}_\lambda \rightarrow W \) surjective?
- We obtain two maps from \( \text{St}_\lambda \rightarrow \Lambda \). Do they agree? In other words, is \( \text{wt}(v) = w, \lambda \)?

**Proposition 4.6.** We have \( \text{wt}(v) = w, \lambda \) and the map \( \text{St}_\lambda \rightarrow W \) is surjective.

**Proof.** Given an element \( w \in W \), we explain how to reconstruct the expected pattern. As the decoration for the pattern comes from the decomposition of \( w \), we only need to specify the nonzero entries. We write \( w = s_{i_1} \cdots s_{i_\ell} \) as in Eq. (5). We reindex the nonzero entries as

\[(b'_1, \ldots, b'_\ell)\]

where \( \ell \) is the length of \( w \).

Each nonzero entry must be the upper bound of its polytope inequality. Therefore, \((b'_1, \ldots, b'_\ell)\) can be defined inductively via

\[
\begin{align*}
b'_\ell &= \langle \lambda, \alpha^\vee_{i_\ell}\rangle \\
b'_{\ell-1} &= \langle \lambda - b'_\ell \alpha^\vee_{i_{\ell-1}}, \alpha^\vee_{i_{\ell-1}}\rangle \\
&\quad \cdots \\
b'_1 &= \langle \lambda - \sum_{k=2}^{\ell} b'_k \alpha^\vee_{i_k}, \alpha^\vee_{i_1}\rangle.
\end{align*}
\]

Now every entry has at least one decoration. Adding zero entries appropriately gives the desired pattern \( \text{BZL}(v_w) \). To verify this is a stable pattern, we need to show that every entry has exactly one decoration.

Before proving that, we find its weight. Given \( w \in W \), we already know a pattern \( \text{BZL}(v_w) \). Let \( v_w \) be the corresponding vertex in \( B_\lambda \). By properties of crystal graph, one can find \( v_w \) by the following path

\[ v_\lambda, f_{i_N}^{b_N}(v_\lambda), f_{i_{N-1}}^{b_{N-1}} f_{i_N}^{b_N}(v_\lambda), \cdots \]

If we would like to ignore the zeros in \( \text{BZL}(v_w) \), we may write \( w = s_{i_1} \cdots s_{i_\ell} \) and one can find \( v_w \) by the following path

\[ v_\lambda, f_{i_\ell}^{b_\ell}(v_\lambda), f_{i_{\ell-1}}^{b_{\ell-1}} f_{i_\ell}^{b_\ell}(v_\lambda), \cdots \]

To avoid burden on notations, we omit the superscript ‘ for the moment.

We claim that

\[ b_j = \langle s_{i_{j+1}} \cdots s_{i_\ell}(\lambda), \alpha^\vee_{i_j}\rangle, \quad \text{wt}(f_{i_j}^{b_j} \cdots f_{s_{i_\ell}}^{b_\ell}(v_\lambda)) = s_{i_j} \cdots s_{i_\ell}(\lambda). \quad (6) \]
We argue by reduction on \( j \). If \( j = \ell \), then this is trivial. Now assume that the result is true for \( j + 1, \ldots, \ell \) and we prove it for \( j \). For a root \( \alpha \),
\[
\langle s_{ij+1} \cdots s_{it}(\lambda), \alpha^\vee \rangle \\
= \langle s_{ij+2} \cdots s_{it}(\lambda), s_{ij+1}(\alpha^\vee) \rangle \\
= \langle s_{ij+2} \cdots s_{it}(\lambda), \alpha^\vee - \langle \alpha^\vee, \alpha_{ij+1} \rangle \alpha_{ij+1}^\vee \rangle \\
= \langle s_{ij+2} \cdots s_{it}(\lambda), \alpha^\vee \rangle - b_{j+1} \langle \alpha^\vee, \alpha_{ij+1} \rangle \\
= \langle s_{ij+3} \cdots s_{it}(\lambda), \alpha^\vee \rangle - b_{j+1} \langle \alpha^\vee, \alpha_{ij+1} \rangle - b_{j+2} \langle \alpha^\vee, \alpha_{ij+2} \rangle \\
\vdots \\
= \langle \lambda - \sum_{k=j+1}^{\ell} b_k \alpha_{ik}, \alpha^\vee \rangle.
\]

Therefore, \( \langle s_{ij+1} \cdots s_{it}(\lambda), \alpha_{ij+1}^\vee \rangle = b_j \). The weight of \( f_{ij}^s \cdots f_{it}^s(v_\lambda) \) is
\[
s_{ij} \cdots s_{it}(\lambda) - b_j \alpha_{ij} = s_{ij+1} \cdots s_{it}(\lambda) - \langle s_{ij+1} \cdots s_{it}(\lambda), \alpha_{ij}^\vee \rangle \alpha_{ij} = s_{ij} \cdots s_{it}(\lambda).
\]

We now lift the assumption on \( \ell \). We verify that BZL\((v_w)\) is a stable pattern. Suppose there is an entry \( b_j \) such that it is boxed and circled. If it is a zero entry, then
\[
0 = b_j = \langle \lambda - \sum_{k=j+1}^{N} b_k \alpha_{ik}, \alpha_{ij}^\vee \rangle.
\]

However, by Eq. (7), the right-hand side is \( \langle s_{ij+1} \cdots s_{it}(\lambda), \alpha_{ij}^\vee \rangle = \langle \lambda, s_{ij} \cdots s_{ij+1} (\alpha_{ij}^\vee) \rangle \). This cannot be zero as we assume \( \langle \lambda, \alpha_{ij}^\vee \rangle > 0 \) for all simple roots. The case \( b_j \neq 0 \) can be analyzed analogously.

\[\square\]

5. Weyl Group Multiple Dirichlet Series

5.1. Definition. Given an isotropic subgroup \( \Omega \) of \( F_S^\times \), let \( \mathcal{M}(\Omega^r) \) be the space of functions \( \Psi : (F_S^\times)^r \to \mathbb{C} \) that satisfy the transformation property
\[
\Psi(\varepsilon c) = \left( \prod_{i=1}^{r} (\varepsilon_i, c_i)_S^{||\alpha_i||^2} \prod_{i<j} (\varepsilon_i, c_j)_S^{2(\alpha_i, \alpha_j)} \right) \Psi(c)
\]
for all \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \Omega^r \) and all \( c = (c_1, \ldots, c_r) \in (F_S^\times)^r \).

Given a root system \( \Phi \) of fixed rank \( r \), and integer \( n \geq 1 \), \( m \in \mathcal{O}_S^r \), and \( \Psi \in \mathcal{M}(\Omega^r) \), we consider a function of \( r \) complex variables \( s = (s_1, \ldots, s_r) \in \mathbb{C}^r \) of the form
\[
Z_\psi(s_1, \ldots, s_r; m_1, \ldots, m_r) = Z_\psi(s; m) = \sum_{c = (c_1, \ldots, c_r) \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r} \frac{H^{(n)}(c; m)\Psi(c)}{|c_1|^{2s_1} \cdots |c_r|^{2s_r}}.
\]

The function \( H(c; m) = H^{(n)}(c; m) \) carries the main arithmetic content. It is not defined as a multiplicative function, but rather a “twisted multiplicative” function. This means that for \( S \)-integer vectors \( c, c' \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r \) with \( \gcd(c_1 \cdots c_r, c'_1 \cdots c'_r) = 1 \),
\[
H(c_1c'_1, \ldots, c_rc'_r; m) = \mu(c, c')H(c; m)H(c'; m),
\]
where \( \mu(c, c') = \prod_{i=1}^{r} \mu(c_i, c'_i) \) is the Möbius function.
where $\mu(c, c')$ is an $n$-th root of unity given by

\[
\mu(c, c') = \prod_{i=1}^{r} \left( \frac{c_i}{\bar{c}_i} \right)^{\| \alpha_i \|^2} \left( \frac{c_i}{\bar{c}_i} \right) \prod_{i<j} \left( \frac{c_i}{\bar{c}_j} \right)^{2\langle \alpha_i, \alpha_j \rangle} \left( \frac{c_j}{\bar{c}_i} \right)^{2\langle \alpha_i, \alpha_j \rangle}.
\]

The transformation property of functions in $\mathcal{M}(\Omega^r)$ implies that

\[
H(ec; m)\Psi(ec) = H(c; m)\Psi(c) \quad \text{for all } e \in \mathcal{O}_S, c, m \in (F_S^r)'\).
\]

The proof requires the $n$-th power reciprocity law ([BBF06] Lemma 1.2).

Now, given any $m, m', c \in \mathcal{O}_S$ with $\gcd(m_1', \cdots, m_r'; c_1 \cdots c_r) = 1$, let

\[
H(c; m_1m_1', \cdots, m_rm_r') = \prod_{i=1}^{r} \left( \frac{m_i'}{c_i} \right)^{-\| \alpha_i \|^2} H(c; m).
\]

Thus, it is enough to specify the coefficients $H(p^k; p^l) := H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for any fixed prime $p$ with $k = (k_1, \cdots, k_r)$, $l = (l_1, \cdots, l_r)$, $k_i = \text{ord}_p(c_i)$ and $l_i = \text{ord}_p(m_i)$ in order to completely determine $H(c; m)$ for any pair of $S$-integer vectors $m$ and $c$.

The goal in the theory of Weyl group multiple Dirichlet series is to find $H(p^k; p^l)$ so that $Z_H(s; m)$ admits meromorphic continuation to $\mathbb{C}^r$ and satisfies a group of functional equations which is isomorphic to $W$.

There are three ways to define $H(p^k; p^l)$. First of all, when $n$ is sufficiently large, the coefficients $H(p^k; p^l)$ admit a Lie-theoretic description ([BBF06, BBF08]). This works for types and all $n$. Secondly, $H(p^k; p^l)$ can be described as a sum over the crystal graph $B_{\lambda+\rho}$, where the contribution is described in terms of $n$-th order Gauss sum and $\lambda = l_1w_1 + \cdots + l_rw_r$. This is proved for type $A$ in [BBF11a], and type $C$ with odd $n$ in [FZ15]. A third approach is given in [CG10], where the generating function for $H(p^k; p^l)$ is constructed. This also works for all root systems and all $n$. However, this description does not yield an explicit formula.

As mentioned in the introduction, these three definitions should agree. In this section, we compare the Lie-theoretic description and the crystal description for type $A$ and type $C$ ($n$ odd). The comparison is given in [BBF08] Section 8 for type $A$, and [BBF11] Section 4 for type $C$ and $n$ odd. In both cases, the proofs rely on explicit realization of the root systems in $\mathbb{R}^r$. In [BBF11] Section 4.1, the authors ask if a coordinate-free proof exists. This is our goal in this section.

5.2. Twisted Weyl group multiple Dirichlet series: the stable case. In this section, we recall the definition of twisted Weyl group multiple Dirichlet series in the stable range in [BBF08].

Fix non-negative integers $l_1, \cdots, l_r$ and let $\lambda = \sum l_iw_i$ be the corresponding weight.

**Stability Assumption.** The positive integer $n$ satisfies the following property. Let $\alpha^\vee = \sum_{i=1}^{r} t_i \alpha_i^\vee$ be the largest positive coroot in the partial ordering. Then for every prime $p$,

\[
n \geq \gcd(n, \| \alpha \|^2) \cdot d_\lambda(\alpha^\vee) = \gcd(n, \| \alpha \|^2) \cdot \sum_{i=1}^{r} t_i(l_i + 1).
\]

**Remark 5.1.** Notice that there is a misprint in the Stability Assumption in [BBF08] Section 3, [BBF11] Section 4, and [FZ15] Section 9. One should use the largest positive coroot instead of the largest positive root in the partial ordering.
Lemma 5.2 ([BBF08], Lemma 1). Let $w \in W$.

(1) There are nonnegative integers $k_i$ such that

$$
\rho + \lambda - w \rho + \lambda = \sum_{i=1}^{r} k_i \alpha_i.
$$

(8)

(2) If $w, w' \in W$ such that $\rho + \lambda - w \rho + \lambda = \rho + \lambda - w' \rho + \lambda$ then $w = w'$.

Recall that by twisted multiplicativity, it remains to describe $H(p^k; p^l)$ for any fixed prime $p$. For any given $(k_1, \cdots, k_r)$, these coefficients are defined to be zero unless there exists $w \in W$ such that Eq. (8) holds. In this case, we define

$$
H(p^k; p^l) = \prod_{\alpha \in \Phi(w)} g_{\|\alpha\|^2}(p^{d_\alpha_p(\alpha)} - 1, p^{d_\alpha_p(\alpha)}).
$$

5.3. Crystal Graph Description. One expect to take the following description

$$
H(p^k; p^l) = \sum_{\nu \in B_{\rho + \lambda}} G(v).
$$

Here the relation between $(k_1, \cdots, k_r)$ and $\mu$ is

$$
\rho + \lambda - w(\mu) = \sum_{i=1}^{r} k_i \alpha_i.
$$

The definition here depends on the choice of a reduced decomposition of $w_0$. Here we choose the nice decompositions in Proposition 3.9.

A definition of $G(v)$ is already given for type $A$ ([BBF11a]) and type $C$ with odd $n$ ([FZ15] Eq. (34)). Again, as explained in Remark 4.1, we take slightly different descriptions to make the comparison more natural.

The definition of $G(v)$ is given as follows:

$$
G(v) = \prod_{b_j \in BZL(v)} \begin{cases}
q^{b_j} & \text{if } b_j \text{ is circled but not boxed}, \\
g_{\|\alpha_j\|^2}(p^{d_\alpha_p(\alpha) - 1}, p^{d_\alpha_p(\alpha)}) & \text{if } b_j \text{ is boxed but not circled}, \\
q^{b_j}(1 - q^{-1}) & \text{if } b_j \text{ is neither circled nor boxed and } n \mid b_j, \\
0 & \text{otherwise}.
\end{cases}
$$

Here, the index $i_j$ is the one appearing in the nice decomposition of $w_0$.

It is still not clear how to define $G(v)$ in general. For Eisenstein series constructing from an automorphic representation on a braidless maximal parabolic subgroup, an analogous calculation has been carried out in [BF15].

5.4. Comparison. In this section, we only work with roots system of type $A$ and type $C$ with odd $n$. We show that in these two cases, the crystal graph description in the stable range agrees with the Lie-theoretic description. We write $H_{BZL}(p^k; p^l)$ for the crystal graph description, and $H_{St}(p^k; p^l)$ for the Lie-theoretic description.

Theorem 5.3. Let $\Phi$ be a root system of type $A$ and $C$. Let $n$ be a positive integer satisfying the stability assumption. We also require $n$ to be odd if $\Phi$ is of type $C$.

(1) If $v \in B_{\rho + \lambda}$ such that $BZL(v)$ is an unstable pattern, then $G(v) = 0$. 


(2) Let $w \in W$. Let $k$ be defined in Eq. (8). Then

$$H_{BZL}(p^k; p^l) = H_{St}(p^k; p^l).$$

Proof. Suppose BZL($v$) is an unstable pattern. If there is an entry which is both circled and boxed, then $G(v) = 0$ from the fourth case in Eq. (9). If one of the entries $b_j$ is neither circled nor boxed, then by Lemma 4.4, $b_j < \langle \lambda, \alpha^*_j \rangle \leq n$. In other words, $n \nmid b_j$. Again, by the fourth case in Eq. (9), we conclude that $G(v) = 0$.

We are now left with the stable patterns. By Proposition 4.6, they are in bijection with $W$. Given $w \in W$, let BZL($v_w$) be the corresponding stable pattern and $v_w$ be the corresponding vertex in $B_{\rho + \lambda}$. Recall that the weight of $v_w$ is $w(\rho + \lambda)$. The pattern BZL($v_w$) contributes to $H_{BZL}(p^k; p^l)$ where $\rho + \lambda - w(\rho + \lambda) = \sum k_i \alpha_i$. This is also the only pattern contributing to this term. On the other hand, $w$ contributes to $H_{St}(p^k; p^l)$ with the same $k$.

To calculate $H_{St}(p^k; p^l)$, we write $w = s_1^i \cdots s_l^i$ as in Eq. (5). Define

$$\beta_\ell = \alpha_\ell^i, \quad \beta_{\ell - 1} = s_\ell^i(\alpha_{\ell - 1}^i), \ldots, \quad \beta_1 = s_\ell^i s_{\ell - 1}^i \cdots s_1^i(\alpha_1^i).$$

Then by Lemma 2.1,

$$\Phi(w) = \{\beta_\ell, \ldots, \beta_1\}.$$

Note that $\|\beta_j\|^2 = 2$ if and only if $\|\alpha_j^i\|^2 = 2$. Thus the contribution of $w$ is

$$H_{St}(p^k; p^l) = \prod_{j=1}^\ell g_{\|\beta_j\|^2}(p^{d_\lambda(\beta_j) - 1}, p^{d_\lambda(\beta_j)}).$$

We now calculate $H_{BZL}(p^k; p^l)$. Write BZL($v_w$) = $(b_1, \cdots, b_N)$. Let $(b_1', \cdots, b_{\ell}')$ be the nonzero entries in BZL($v_w$). By Eq. (6),

$$(b_1', \cdots, b_{\ell}') = (d_\lambda(\beta_1), \cdots, d_\lambda(\beta_{\ell})).$$

Note that the circled entries are zero. Thus

$$H_{BZL}(p^k; p^l) = G(v_w) = \prod_{b_j \in \text{BZL}(v_w), b_j \neq 0} g_{\|\alpha_j\|^2}(p^{b_j - 1}, p^{b_j}) = \prod_{j=1}^\ell g_{\|\alpha_j^i\|^2}(p^{b_j' - 1}, p^{b_j'}),$$

where the index $i_j$ is the one appearing in the nice decomposition of $w_0$. The proof is completed.

\[\square\]

**APPENDIX A. EXAMPLES**

In this appendix, we work out $T_{\tau_\ell}$ (see Remark 3.8) and the bijection in Corollary 3.7 for all the braidless weights.

**A.1. Type A.** Let $g = s_1 r_+ 1$ and $\omega = \omega_1$. Then $\tau_\ell = s_1 s_2 \cdots s_r \in \omega W$. The graph $T_{\tau_\ell}$ is

$$s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_r.$$

and $\omega W = \{I, s_1, s_1 s_2, \cdots, s_1 s_2 \cdots s_r\}$.

If we pick a general $\omega = \omega_j$, then

$$\tau_\ell = (s_j s_{j-1} \cdots s_1)(s_{j+1} s_1 \cdots s_2) \cdots (s_r s_{r-1} \cdots s_{r-j+1}).$$
The graph $T_{\tau_l}$ is

```
  s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_{r-j+1}
  \uparrow \quad \uparrow \quad \uparrow
  \vdots \quad \vdots 
  \uparrow \quad \uparrow 
  s_{j-1} \rightarrow s_j \rightarrow \cdots \rightarrow s_{r-1}
  \uparrow \quad \uparrow \quad \uparrow
  s_j \rightarrow s_{j+1} \rightarrow \cdots \rightarrow s_r
```

Using Corollary 3.7, it is easy to show that there is a bijection between

$$
\{(i_j, \cdots, i_k, 0, \cdots, 0) \in \mathbb{Z}^{r-j+1} : j-1 \leq k \leq r, 1 \leq i_j < i_{j+1} < \cdots < i_k \leq k\}
$$

and $\omega W$. (When $k = j-1$, we only consider the element $(0,0,\cdots,0) \in \mathbb{Z}^{r-j+1}$.) The map is given by

$$(i_j, \cdots, i_k, 0, \cdots, 0) \mapsto (s_js_{j-1} \cdots s_i)(s_{j+1}s_j \cdots s_{i+1}) \cdots (s_ks_{k-1} \cdots s_{i_k}).$$

(See also [Caiar] Theorem 2.4 for a description in terms of Young tableaux.)

A.2. Type B, C. For $\Phi = B_r$ or $C_r$, the good enumeration in [Lit98] is given as follows:

$$\alpha_1 = \alpha_2 - \alpha_3 - \cdots - \alpha_r.$$

If $\omega = \omega_r$, then

$$\tau_l = s_rs_{r-1} \cdots s_2s_1.$$

The graph $T_{\tau_l}$ is

```
  s_r \rightarrow s_{r-1} \rightarrow \cdots \rightarrow s_2 \rightarrow s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_{r-1} \rightarrow s_r
```

and $\omega W = \{I, s_r, s_rs_{r-1}, \cdots, s_rs_{r-1} \cdots s_1, s_rs_{r-1} \cdots s_1s_2, \cdots, \tau_l\}$. If $\omega = \omega_1$, then

$$\tau_l = (s_1s_2 \cdots s_r)(s_1s_2 \cdots s_{r-1}) \cdots (s_1).$$

The graph $T_{\tau_l}$ is

```
  s_r \rightarrow s_{r-1} \rightarrow \cdots \rightarrow s_2 \rightarrow s_1 
  \uparrow \quad \uparrow \quad \uparrow
  \vdots \quad \vdots 
  \uparrow \quad \uparrow 
  s_3 \rightarrow s_2 \rightarrow s_1 
  \uparrow \quad \uparrow 
  s_2 \rightarrow s_1 
  \uparrow 
  s_1
```

There is a bijection

$$\{(i_1, \cdots, i_k, 0, \cdots, 0) \in \mathbb{Z}^r : 0 \leq k \leq r, r \geq i_1 > \cdots > i_k > 0\} \rightarrow \omega W,$$

where the map is given by

$$(i_1, \cdots, i_k, 0, \cdots, 0) \mapsto (s_1s_2 \cdots s_{i_1})(s_1s_2 \cdots s_{i_2}) \cdots (s_1s_2 \cdots s_{i_k}).$$
Remark A.1. This parametrization is slightly different from the parametrization in [GPSR87] page 20. With our parametrization, one can show that

\[
\Phi(w) = \bigcup_{l=1}^{k} \Phi(w, l),
\]

where \(\Phi(w, l)\) is the set of roots corresponding to the first \(i_l\) roots (counting from the bottom) in the \((r+l)\)th column of \(Sp_{2r}\). This parametrization may simplify the calculation in [GPSR87] Section 5.

A.3. Type D. The enumeration in [Lit98] is given as follows:

\[
\begin{array}{c}
\alpha_1 \\
\alpha_3 - \cdots - \alpha_r \\
\alpha_2
\end{array}
\]

If \(\omega = \omega_r\), then \(\tau_\ell = s_r s_{r-1} \cdots s_3 s_1 s_{2} s_{3} \cdots s_{r-1} s_r\). The graph \(T_{\tau_\ell}\) is

\[
\begin{array}{c}
s_1 \\
s_2 \searrow \\
\vdots \searrow \cdots \searrow \downarrow \\
1 \searrow \downarrow \\
s_4 \searrow \downarrow \cdots \searrow \downarrow \\
s_3 \searrow \downarrow \downarrow \\
s_2 \searrow \downarrow \downarrow \\
s_1
\end{array}
\]

and

\[
\omega W = \{I, s_r, s_r s_{r-1}, \cdots, s_r s_{r-1} \cdots s_3 s_1, s_r s_{r-1} \cdots s_3 s_2, s_r s_{r-1} \cdots s_3 s_1 s_2, s_r s_{r-1} \cdots s_3 s_1 s_2 s_3, \cdots, \tau_\ell\}.
\]

If \(\omega = \omega_1\) (the case \(\omega = \omega_2\) is similar), then

\[
\tau_\ell = (s_1 s_3 s_4 \cdots s_r)(s_2 s_3 \cdots s_{r-1})(s_1 s_3 \cdots s_{r-2})(s_1 s_3 \cdots s_{r-3}) \cdots (s_1).
\]

The graph \(T_{\tau_\ell}\) is

\[
\begin{array}{c}
s_r \searrow s_{r-1} \cdots \searrow s_3 \searrow s_1 \\
1 \searrow \downarrow \\
\vdots \searrow \cdots \searrow s_1 \\
1 \searrow \downarrow \\
s_4 \searrow \downarrow \cdots \searrow \downarrow \\
s_3 \searrow \downarrow \downarrow \\
s_2 \searrow \downarrow \downarrow \\
s_1
\end{array}
\]

There is a bijection between

\[
\{(i_1, \cdots, i_k, 0, \cdots, 0) \in \mathbb{Z}^{r-1} : 0 \leq k \leq r-1, r \geq i_1 > \cdots > i_k > 0, i_k \neq 2 \text{ if } k \neq 2, i_2 \neq 1\}
\]

and \(\omega W\), where the map is given by

\[
(i_1, \cdots, i_k, 0, \cdots, 0) \mapsto (s_1 s_3 s_4 \cdots s_{i_1})(s_2 s_3 \cdots s_{i_2})(s_1 s_3 \cdots s_{i_3}) \cdots (s_1 s_3 \cdots s_{i_k}).
\]

The remark in the previous section applies in this case as well.
A.4. **Type E.** For $E_6$ and $E_7$, the good enumerations in [Lit98] are given as follows:

For $E_6$ and $E_7$, the good enumerations in [Lit98] are given as follows:

$$
\begin{array}{c}
\alpha_1 \\
\alpha_5 - \alpha_4 - \alpha_3 - \alpha_2 - \alpha_6 \\
\end{array}
\quad
\begin{array}{c}
\alpha_1 \\
\alpha_5 - \alpha_4 - \alpha_3 - \alpha_2 - \alpha_6 - \alpha_7 \\
\end{array}
$$

When $\Phi = E_6$, we choose $\omega = \omega_6$. When $\Phi = E_7$, we choose $\omega = \omega_7$. The longest elements $\tau_\ell$ are

$$
E_6 : s_6 s_2 s_3 s_1 s_4 s_5 s_3 s_4 s_2 s_3 s_1 s_6 s_2 s_3 s_4 s_5 \\
E_7 : s_7 s_6 s_2 s_3 s_1 s_4 s_5 s_3 s_4 s_2 s_3 s_1 s_6 s_2 s_3 s_4 s_5 s_7 s_6 s_2 s_3 s_1 s_4 s_3 s_2 s_6 s_7.
$$

Here are the graphs:

$$
\begin{array}{c}
E_7 \\
s_7 \\
\uparrow \\
s_6 \\
\uparrow \\
s_2 \\
\uparrow \\
s_1 \rightarrow s_3 \\
\uparrow \\
s_7 \rightarrow s_6 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_5
\end{array}
$$

$$
\begin{array}{c}
E_6 \\
s_6 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_5 \\
\uparrow \\
s_2 \rightarrow s_3 \rightarrow s_1 \\
\uparrow \\
s_1 \rightarrow s_3 \rightarrow s_4 \\
\uparrow \\
s_6 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_5,
\end{array}
$$

With a small calculation, one can write down all 27 (56, resp.) minimal representatives.

A.5. **Type G.** This is relatively easier. We include it here for completeness.

Any enumeration is a good enumeration for $G_2$. Let $\Delta = \{\alpha_1, \alpha_2\}$ be a set of simple roots for $G_2$, where $\alpha_2$ is the long root. If $\omega = \omega_1$, then the longest element is $\tau_\ell = s_1 s_2 s_1 s_2 s_1$ and the graph $T_{\tau_\ell}$ is

$$
\begin{array}{c}
s_1 \rightarrow s_2 \rightarrow s_1 \rightarrow s_2 \rightarrow s_1
\end{array}
$$

If $\omega = \omega_2$, then the longest element is $\tau_\ell = s_2 s_1 s_2 s_1 s_2$ and the graph $T_{\tau_\ell}$ is

$$
\begin{array}{c}
s_2 \rightarrow s_1 \rightarrow s_2 \rightarrow s_1 \rightarrow s_2
\end{array}
$$

References
