1. Preliminaries

A Lie algebra is a vector space $g$ over a field $k$, with a bilinear operation $g \times g \rightarrow g$, denoted $(x, y) \mapsto [x, y]$ and called the bracket or commutator of $x$ and $y$, which satisfies the following axioms, for all $x, y, z \in g$:

1. $[x, x] = 0$ (anticommutativity),
2. $[x[yz]] + [y[zx]] + [z[xy]] = 0$ (Jacobi identity).

Note that property (1) implies that $[x, y] = -[y, x]$ for all $x, y \in g$. If $\text{char} \ k \neq 2$, then this condition is equivalent to (1).

Example 1.1. Let $A$ be an associative algebra with bilinear operation denoted $x \cdot y$ for $x, y \in A$. We can define a new operation $[-, -]$, called the bracket of $x$ and $y$, as follows:

$$[x, y] = x \cdot y - y \cdot x.$$ 

Then $A$ with the operation $[-, -]$ is a Lie algebra.

Example 1.2. The general linear algebra. Let $V$ be a finite dimensional vector space over $k$, and denote by $\text{End}(V)$ the set of linear transformations $V \rightarrow V$. Then $\text{End}(V)$ with the bracket operation $[-, -]$ is a Lie algebra, which we write as $\mathfrak{gl}(V)$. If we choose a basis for $V$, we may identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices over $k$, and we denote this by $\mathfrak{gl}_n(k)$. The dimension of $\mathfrak{gl}_n(k)$ is $n^2$.

Example 1.3. The special linear algebra. Define $\mathfrak{sl}_n(k) = \{ x \in \mathfrak{gl}_n \mid \text{Tr}(x) = 0 \}$, where $\text{Tr}(x)$ is the sum of the diagonal elements of the matrix $x$.

Example 1.4.

$$\mathfrak{sl}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in k \right\}$$

has as a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Then one can check that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$ 

A vector subspace $B \subseteq g$ is called a subalgebra if $a, b \in B$ implies $[a, b] \in B$. A vector subspace $I \subseteq g$ is called an ideal if $a \in A, x \in I$ imply $[a, x], [x, a] \in I$. We call an a Lie algebra $g$ simple if $\dim(g) \geq 2$ and $g$ has no non-trivial ideals.
A linear map $f : g \to a$ of Lie algebras is called a homomorphism if $f([a, b]) = [f(a), f(b)]$ for each $a, b \in g$. A representation of a Lie algebra $g$ is a homomorphism $\phi : g \to \mathfrak{gl}(V)$. A representation is called faithful if the kernel of $\phi$ is trivial. The dimension of a representation is by definition the dimension of the vector space $V$.

Then linear endomorphism $d : g \to g$ is called a derivation if the following Leibniz rule holds: $d([a, b]) = [d(a), b] + [a, d(b)]$. Define a linear transformation $ad_x : g \to g$ by the formula $ad_x(y) = [x, y]$. Then $ad_x$ is a derivation by the Jacobi identity. Derivations of the form $ad_x$ are called inner derivations, and all others are called outer derivations.

Let $g$ be a Lie algebra, and define a map $ad : g \to \mathfrak{gl}(g)$ by $x \mapsto ad_x$. (For the definition of $\mathfrak{gl}(g)$ one considers $g$ only as a vector space.) The map $ad : g \to \mathfrak{gl}(g)$ is a homomorphism of Lie algebras, and is refereed to as the adjoint representation of $g$.

2. SIMPLE FINITE-DIMENSIONAL LIE ALGEBRAS

Let $g$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra $h$. All Cartan subalgebras are conjugate. Define the rank of $g$ to be $\operatorname{rk} g := \dim h$. For $\alpha \in h^*$, let

$$g_\alpha := \{ x \in g \mid [h, x] = \alpha(h)x \text{ for all } h \in h \}$$

and let $\Delta = \{ \alpha \in h^* \setminus \{0\} \mid g_\alpha \neq 0 \}$. Then $g_0 = h$ and $g$ has a root space decomposition

$$g = h \oplus \bigoplus_{\alpha \in \Delta} g_\alpha.$$

Note that for $\alpha \in \Delta$, $\dim g_\alpha = 1$. Note that $[g_\alpha, g_\beta] \subseteq g_{\alpha + \beta}$. There is a nondegenerate invariant symmetric bilinear form $(\cdot, \cdot)$ on $g$. Any such form is proportional to the Killing form $\kappa(x, y) := \operatorname{Trace}((\operatorname{ad} x)(\operatorname{ad} y))$. The restriction of the form to $h$ is nondegenerate.

We fix a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \Delta$. Then for each $\beta \in \Delta$ there are integers $k_i$ such that $\beta = \sum_{i=1}^n k_i \alpha_i$. Moreover, $\Delta$ is the disjoint union of $\Delta_+ := \{\beta \in \Delta \mid k_i \geq 0\}$ and $\Delta_- := \{\beta \in \Delta \mid k_i \leq 0\}$. Note that $\Pi$ is a basis for the vector space $h^*$, so in particular $n = \operatorname{rk} g$. The decomposition of $\Delta$ determines a decomposition $g = n^- \oplus h \oplus n^+$ where $n^+ := \oplus_{\alpha \in \Delta^+} g_\alpha$ and $n^- := \oplus_{\alpha \in \Delta^-} g_\alpha$. This is called a triangular decomposition.

**Example 2.1.** Roots for $g = \mathfrak{sl}(n)$. Let $h$ be the set of traceless $n \times n$-diagonal matrices. Then $h$ is a Cartan subalgebra for $\mathfrak{sl}(n)$. For $i = 1, \ldots, n$, define $\varepsilon_i : h \to \mathbb{C}$ by $\varepsilon_i(h) = h_{ii}$ (the $i^{th}$ diagonal entry of $h$). Then for $i, j \in \{1, \ldots, n\}$, $i \neq j$, we have $g_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$. Since the $E_{ij}$ : $i \neq j$ together with $h$ span $g$, we conclude that $\Delta = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq n}$ and $\dim g_{\varepsilon_i - \varepsilon_j} = 1$.

Let $h^* = \{\text{linear maps } f : h \to \mathbb{C}\}$. We can identify $h$ with $h^*$ as follows. Define a linear map $\nu : h \to h^*$ by $\nu(h)(h') = (h, h')$. Then one can define $(\cdot, \cdot)$ on $h^*$ using this identification. Now for $\alpha \in h^*$ let $\alpha' := \frac{2\nu^{-1}(\alpha)}{\langle \alpha, \alpha \rangle}$. The Cartan matrix $A$ is defined to be the $n \times n$ matrix with entries $a_{ij} := \langle \alpha_i', \alpha_j \rangle$. A Dynkin diagram encodes the information of the Cartan matrix in a diagram.
Example 2.2. Cartan matrix and Dynkin diagram for \( \mathfrak{sl}(5) \).

\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\]

\[\bigcirc - \bigcirc - \bigcirc - \bigcirc \]

Let \( \mathfrak{g} \) be a simple finite-dimensional Lie algebra and let \( \mathfrak{h} \) be a Cartan subalgebra. Fix a base \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) for the corresponding root system \( \Delta \). Let \( A \) be the Cartan matrix. For each \( i = 1, \ldots, n \), choose \( e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i} \) such that \( h_i := [e_i, f_i] \) satisfy the \( \mathfrak{sl}_2 \) relations given in Example 1.4. Then \( \mathfrak{n}^+ \) is generated by the elements \( e_i \) and \( \mathfrak{n}^- \) is generated by the elements \( f_i \). So \( \mathfrak{g} \) is generated by the elements \( e_i, f_i, h_i \) with \( 1 \leq i \leq n \). These are called the Chevalley generators. These elements satisfy the Weyl relations. For \( 1 \leq i, j \leq n \):

1. \( [h_i, h_j] = 0; \)
2. \( [e_i, f_j] = \delta_{ij}h_i; \)
3. \( [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j. \)

They also satisfy the Serre relations:

1. \( \text{ad}(e_i)^{-a_{ij}+1}e_j = 0 \) (\( i \neq j \));
2. \( \text{ad}(f_i)^{-a_{ij}+1}f_j = 0 \) (\( i \neq j \)).

Moreover, \( \mathfrak{g} \) is defined by these generators and relations.

### 3. Affine Lie algebras

Let \( \mathfrak{g} \) be a simple finite-dimensional Lie algebra and let \( (\cdot, \cdot) \) be a non-degenerate invariant symmetric bilinear form. The associated (non-twisted) affine Lie superalgebra is

\[
\hat{\mathfrak{g}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d
\]

with commutation relations

\[
[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m, -n}(a, b)K, \quad [K, \hat{\mathfrak{g}}] = 0 \quad [d, at^m] = mat^m, \quad [d, d] = 0
\]

where \( a, b \in \mathfrak{g}; m, n \in \mathbb{Z} \). Note that \( (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \) is a central extension of \( (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \), and \( d \) is an outer derivation of \( (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \). By identifying \( \mathfrak{g} \) with \( 1 \otimes \mathfrak{g} \), we have that the Cartan subalgebra of \( \hat{\mathfrak{g}} \) is

\[
\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d
\]

The set of roots \( \hat{\Delta} \) for \( \hat{\mathfrak{g}} \) is calculated as follows. For each \( \alpha \in \Delta \subset \hat{\mathfrak{h}}^* \), we can extend \( \alpha \) to \( \hat{\mathfrak{h}} \) by letting \( \alpha(K) = \alpha(d) = 0 \). Let \( \delta \) be the linear function defined on \( \mathfrak{h} \) by \( \delta |_{\mathfrak{h} \otimes \mathbb{C}K} = 0 \) and \( \delta(d) = 1 \). For each \( \alpha \in \Delta \), fix \( x_\alpha \in \mathfrak{g}_\alpha \) nonzero. Then for each \( \alpha \in \Delta \), \( m \in \mathbb{Z} \) one can check by direct calculation that \( x_\alpha t^m \in \hat{\mathfrak{g}}_{m\delta + \alpha} \). Similarly, one can check that for each \( h \in \mathfrak{h} \) and \( m \in \mathbb{Z} \setminus \{0\} \) that one has \( h t^m \in \hat{\mathfrak{g}}_{m\delta} \). If we fix a basis \( h_1, \ldots, h_n \) for \( \mathfrak{h} \), then the following is a basis for \( \hat{\mathfrak{g}} \):

\[
\{x_\alpha t^m\}_{m \in \mathbb{Z}, \alpha \in \Delta} \cup \{h_i t^m\}_{m \in \mathbb{Z} \setminus \{0\}, i = 1, \ldots, n} \cup \{h_1, \ldots, h_n, K, d\}.
\]

In particular,

\[
\hat{\Delta} = \{m\delta + \alpha\}_{m \in \mathbb{Z}, \alpha \in \Delta} \cup \{m\delta\}_{m \in \mathbb{Z} \setminus \{0\}},
\]

\[\dim \hat{\mathfrak{g}}_{m\delta + \alpha} = 1, \text{ and } \dim \hat{\mathfrak{g}}_{m\delta} = \text{rk } \mathfrak{g}.\]