Introduction to affine Lie algebras
Extended notes for Lecture 1
October 19, 2010
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1. Preliminaries

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field k, with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(x, y) \mapsto [x, y]$ and called the *bracket* or *commutator* of x and y, which satisfies the following axioms, for all $x, y, z \in \mathfrak{g}$:

(1) $[x, x] = 0$ (anticommutativity),
(2) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ (Jacobi identity).

Note that property (1) implies that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. If $\operatorname{char} k \neq 2$, then this condition is equivalent to (1).

Example 1.1. Let A be an associative algebra with bilinear operation denoted $x \cdot y$ for $x, y \in A$. We can define a new operation $[-, -]$,called the bracket of x and y, as follows:

$$[x, y] = x \cdot y - y \cdot x.$$

Then A with the operation $[-, -]$ is a Lie algebra.

Example 1.2. The general linear algebra. Let V be a finite dimensional vector space over k, and denote by $\operatorname{End}(V)$ the set of linear transformations $V \to V$. Then $\operatorname{End}(V)$ with the bracket operation $[-, -]$ is a Lie algebra, which we write as $\mathfrak{gl}(V)$. If we choose a basis for V, we may identify $\mathfrak{gl}(V)$ with the set of $n \times n$ matrices over k, and we denote this by $\mathfrak{gl}_n(k)$. The dimension of $\mathfrak{gl}_n(k)$ is n^2.

Example 1.3. The special linear algebra. Define $\mathfrak{sl}_n(k) = \{x \in \mathfrak{gl}_n \mid \operatorname{Tr}(x) = 0\}$, where $\operatorname{Tr}(x)$ is the sum of the diagonal elements of the matrix x.

Example 1.4.

$$\mathfrak{sl}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in k \right\}$$

has as a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then one can check that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

A vector subspace $B \subseteq \mathfrak{g}$ is called a *subalgebra* if $a, b \in B$ implies $[a, b] \in B$. A vector subspace $I \subseteq \mathfrak{g}$ is called an *ideal* if $a \in A$, $x \in I$ imply $[a, x]$, $[x, a] \in I$. We call an a Lie algebra $\mathfrak{g}$ *simple* if $\dim(\mathfrak{g}) \geq 2$ and $\mathfrak{g}$ has no non-trivial ideals.

A linear map $f : \mathfrak{g} \to \mathfrak{a}$ of Lie algebras is called a *homomorphism* if $f([a,b]) = [f(a), f(b)]$ for each $a, b \in \mathfrak{g}$. A *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. A representation is called *faithful* if the kernel of ϕ is trivial. The dimension of a representation is by definition the dimension of the vector space V.

Then linear endomorphism $d : \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* if the following *Leibniz rule* holds: $d([a,b]) = [d(a), b] + [a, d(b)]$. Define a linear transformation $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by the formula $\mathrm{ad}_x(y) = [x,y]$. Then ad_x is a derivation by the Jacobi identity. Derivations of the form ad_x are called *inner derivations*, and all others are called *outer derivations*.

Let $\mathfrak{g}$ be a Lie algebra, and define a map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $x \mapsto \mathrm{ad}_x$. (For the definition of $\mathfrak{gl}(\mathfrak{g})$ one considers $\mathfrak{g}$ only as a vector space.) The map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras, and is refereed to as the adjoint representation of $\mathfrak{g}$.

2. Simple finite-dimensional Lie algebras

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}$. All Cartan subalgebras are conjugate. Define the rank of $\mathfrak{g}$ to be $\mathrm{rk}\ \mathfrak{g} := \dim \mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \ \text{ for all } h \in \mathfrak{h}\}$$

and let $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$. Then $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Note that for $\alpha \in \Delta$, $\dim \mathfrak{g}_\alpha = 1$. Note that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. There is a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. Any such form is proportional to the Killing form $\kappa(x,y) := \mathrm{Trace}((\mathrm{ad}\ x)(\mathrm{ad}\ y))$. The restriction of the form to $\mathfrak{h}$ is nondegenerate.

We fix a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \Delta$. Then for each $\beta \in \Delta$ there are integers k_i such that $\beta = \sum_{i=1}^n k_i \alpha_i$. Moreover, Δ is the disjoint union of $\Delta_+ := \{\beta \in \Delta \mid k_i \geq 0\}$ and $\Delta_- := \{\beta \in \Delta \mid k_i \leq 0\}$. Note that Π is a basis for the vector space $\mathfrak{h}^*$, so in particular $n = \mathrm{rk}\ \mathfrak{g}$. The decomposition of Δ determines a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^+ := \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- := \oplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. This is called a *triangular decomposition*.

Example 2.1. Roots for $\mathfrak{g} = \mathfrak{sl}(n)$. Let $\mathfrak{h}$ be the set of traceless $n \times n$-diagonal matrices. Then $\mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{sl}(n)$. For $i = 1, \ldots, n$, define $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$ (the i^{th} diagonal entry of h). Then for $i, j \in \{1, \ldots, n\}$, $i \neq j$, we have $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}$. Since the $E_{ij} : i \neq j$ together with $\mathfrak{h}$ span $\mathfrak{g}$, we conclude that $\Delta = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq n}$ and $\dim \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = 1$.

Let $\mathfrak{h}^* = \{\text{linear maps } f : \mathfrak{h} \to \mathbb{C}\}$. We can identify $\mathfrak{h}$ with $\mathfrak{h}^*$ as follows. Define a linear map $\nu : \mathfrak{h} \to \mathfrak{h}^*$ by $\langle \nu(h), h' \rangle = (h, h')$. Then one can define $(\cdot,\cdot)$ on $\mathfrak{h}^*$ using this identification. Now for $\alpha \in \mathfrak{h}^*$ let $\alpha^\vee := \frac{2\nu^{-1}(\alpha)}{(\alpha,\alpha)}$. The *Cartan matrix* A is defined to be the $n \times n$ matrix with entries $a_{ij} := \langle \alpha_i^\vee, \alpha_j \rangle$. A *Dynkin diagram* encodes the information of the Cartan matrix in a diagram.

Example 2.2. Cartan matrix and Dynkin diagram for $\mathfrak{sl}(5)$.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad\qquad \bigcirc - \bigcirc - \bigcirc - \bigcirc$$

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Fix a base $\Pi = \{\alpha_1, \dots, \alpha_n\}$ for the corresponding root system Δ. Let A be the Cartan matrix. For each $i = 1, \dots, n$, choose $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $h_i := [e_i, f_i]$ satisfy the sl_2 relations given in Example 1.4. Then $\mathfrak{n}^+$ is generated by the elements e_i and $\mathfrak{n}^-$ is generated by the elements f_i. So $\mathfrak{g}$ is generated by the elements e_i, f_i, h_i with $1 \leq i \leq n$. These are called the Chevalley generators. These elements satisfy the Weyl relations. For $1 \leq i, j \leq n$:

(1) $[h_i, h_j] = 0$;
(2) $[e_i, f_j] = \delta_{ij} h_i$;
(3) $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$.

They also satisfy the Serre relations:

(1) $\mathrm{ad}(e_i)^{-a_{ij}+1} e_j = 0 \qquad (i \neq j)$;
(2) $\mathrm{ad}(f_i)^{-a_{ij}+1} f_j = 0 \qquad (i \neq j)$.

Moreover, $\mathfrak{g}$ is defined by these generators and relations.

3. Affine Lie algebras

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and let $(\cdot, \cdot)$ be a nondegenerate invariant symmetric bilinear form. The associated (non-twisted) affine Lie superalgebra is

$$\widehat{\mathfrak{g}} = \left(\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}\right) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with commutation relations

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a, b)K, \quad [K, \widehat{\mathfrak{g}}] = 0 \quad [d, at^m] = mat^m, \quad [d, d] = 0$$

where $a, b \in \mathfrak{g}$; $m, n \in \mathbb{Z}$. Note that $(\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}K$ is a central extension of $(\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g})$, and d is an outer derivation of $(\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}K$. By identifying $\mathfrak{g}$ with $1 \otimes \mathfrak{g}$, we have that the Cartan subalgebra of $\widehat{\mathfrak{g}}$ is

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

. The set of roots $\widehat{\Delta}$ for $\widehat{\mathfrak{g}}$ is calculated as follows. For each $\alpha \in \Delta \subset \mathfrak{h}^*$, we can extend α to $\widehat{\mathfrak{h}}$ by letting $\alpha(K) = \alpha(d) = 0$. Let δ be the linear function defined on $\mathfrak{h}$ by $\delta \mid_{\mathfrak{h} \oplus \mathbb{C}K} = 0$ and $\delta(d) = 1$. For each $\alpha \in \Delta$, fix $x_\alpha \in \mathfrak{g}_\alpha$ nonzero. Then for each $\alpha \in \Delta$, $m \in \mathbb{Z}$ one can check by direct calculation that $x_\alpha t^m \in \widehat{\mathfrak{g}}_{m\delta+\alpha}$. Similarly, one can check that for each $h \in \mathfrak{h}$ and $m \in \mathbb{Z} \setminus \{0\}$ that one has $ht^m \in \widehat{\mathfrak{g}}_{m\delta}$. If we fix a basis $h_1, \dots, h_n$ for $\mathfrak{h}$, then the following is a basis for $\widehat{\mathfrak{g}}$:

$$\{x_\alpha t^m\}_{m \in \mathbb{Z}, \alpha \in \Delta} \cup \{h_i t^m\}_{m \in \mathbb{Z} \setminus \{0\}, i=1,\dots,n} \cup \{h_1, \dots, h_n, K, d\}.$$

In particular,

$$\widehat{\Delta} = \{m\delta + \alpha\}_{m \in \mathbb{Z}, \alpha \in \Delta} \cup \{m\delta\}_{m \in \mathbb{Z} \setminus \{0\}},$$

$\dim \widehat{\mathfrak{g}}_{m\delta+\alpha} = 1$, and $\dim \widehat{\mathfrak{g}}_{m\delta} = \mathrm{rk}\ \mathfrak{g}$.