Introduction to affine Lie algebras
Notes for Lecture 2
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Crystal Hoyt

1. Preliminaries

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (over $\mathbb{C}$). Fix a set of simple roots. $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \Delta$. Let $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the corresponding coroots.

We can define a partial ordering on Δ as follows. For $\alpha, \beta \in \Delta$, $\alpha > \beta$ if and only if $\alpha - \beta \in Q_+ := \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$. Let θ be the highest root for Δ_+ (defined by this ordering). Then θ is uniquely determine by Π since $\mathfrak{g}$ is simple. Now $\mathfrak{g}$ has a non-degenerate invariant symmetric bilinear form $(\cdot,\cdot)$, which is uniquely determined up to a scalar. We normalize this form by assuming that $(\theta, \theta) = 2$.

2. Simple roots and the Cartan matrix for $\widehat{\mathfrak{g}}$

As a vector space $\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We can extend $(\cdot,\cdot)$ to $\mathfrak{g}$ as follows. For all $m, n \in \mathbb{Z}$ and $a, b \in \mathfrak{g}$. Let

$$(at^m, bt^n) = \delta_{m,-n}(a,b) \qquad (K,K) = (d,d) = 0$$
$$(K,d) = 1 \qquad (at^m, K) = (at^m, d) = 0.$$

Recall from the last lecture that $\widehat{\Delta} = \{m\delta + \alpha\}_{m\in\mathbb{Z},\alpha\in\Delta} \cup \{m\delta\}_{m\in\mathbb{Z}\setminus\{0\}}$. If Π is a set of simple roots for $\mathfrak{g}$ with highest root θ, then

$$\widehat{\Pi} := \{\delta - \theta\} \cup \Pi$$

is a set of simple roots for $\widehat{\mathfrak{g}}$. We define $\alpha_0 := \delta - \theta$. One can check that

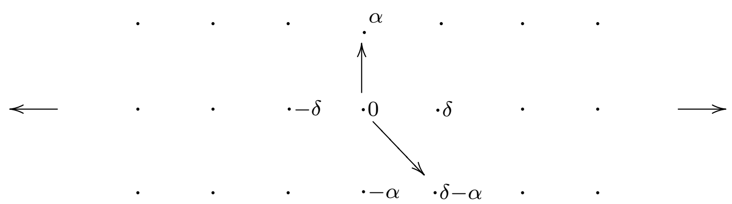
$$\widehat{\Delta}_+ = \{m\delta + \alpha\}_{m\in\mathbb{Z}_{>0},\alpha\in\Delta} \cup \{m\delta\}_{m\in\mathbb{Z}_{>0}} \cup \Delta_+.$$

Corresponding to the decomposition $\widehat{\Delta} = \widehat{\Delta}_+ \cup \widehat{\Delta}_+$ we have a triangular decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-.$$

Example 2.1. $\widehat{\mathfrak{sl}}(2)$. Let $\Delta = \{\pm\alpha\}$ be the roots of $\mathfrak{sl}(2)$ and $\Pi = \{\alpha\}$. Then

$$\widehat{\Delta} = \{m\delta \pm \alpha\}_{m\in\mathbb{Z}} \cup \{m\delta\}_{m\in\mathbb{Z}\setminus\{0\}}, \quad \text{and} \quad \widehat{\Pi} = \{\delta - \alpha, \alpha\}.$$



Let $\{e, f, h\}$ be a basis satisfying the relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$. Set

$$e_0 = ft \qquad \alpha_0 = \delta - \alpha$$
$$f_0 = et^{-1} \qquad \alpha_0^\vee = K - h.$$

Then e_0, e generate $\widehat{\mathfrak{n}}_+$, and f_0, f generate $\widehat{\mathfrak{n}}_-$.

The Cartan matrix and Dynkin diagram of $\widehat{\mathfrak{sl}}(2)$ are

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad\qquad \bigcirc \Longleftrightarrow \bigcirc\,.$$

Now we look at the general case. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the Chevalley involution, which is defined by $e_i \mapsto -f_i : i = 1, \ldots, n$ and $h \mapsto -h$ for all $h \in \mathfrak{h}$. Take $f_\theta \in \mathfrak{g}_{-\theta}$ such that $(-\omega(f_\theta), f_\theta) = 1$. Let $e_\theta = -\omega(f_\theta)$. Then $[e_\theta, f_\theta] = \theta^\vee$. Set

$$\begin{aligned} e_0 &= f_\theta t & \qquad\qquad \alpha_0 &= \delta - \theta \\ f_0 &= e_\theta t^{-1} & \alpha_0^\vee &= K - \theta^\vee. \end{aligned}$$

For $i = 1, \ldots, n$, choose $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $[e_i, f_i] = \alpha_i^\vee$. Then $e_0, e_1, \ldots, e_n$ generate $\widehat{\mathfrak{n}}_+$, and $f_0, f_1, \ldots, f_n$ generate $\widehat{\mathfrak{n}}_-$.

The Cartan matrix A for $\widehat{\mathfrak{g}}$ is obtained from the Cartan matrix A' for $\mathfrak{g}$ by adding one row and column. Explicitly, $a_{i0} = \langle \alpha_i^\vee, \alpha_0 \rangle = -\langle \alpha_i^\vee, \theta \rangle$ and $a_{0i} = \langle \alpha_0^\vee, \alpha_i \rangle = -\langle \alpha_i, \theta^\vee \rangle$. Now A' is invertible and the corank of A is 1.

3. Kac-Moody algebras

An $n \times n$-matrix is called a *generalized Cartan matrix* if for $i, j = 1, \ldots, n$,

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Let A be a generalized Cartan matrix, and let $\mathfrak{h}$ be a vector space (over $\mathbb{C}$) with dimension $n + \text{corank}(A)$. Let

$$\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$$
$$\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$$

be linearly independent sets satisfying

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}.$$

Define $\bar{\mathfrak{g}}(A)$ by generators: $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, $\mathfrak{h}$ and relations:

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} \alpha_i^\vee & \qquad\qquad [h, e_i] &= \langle h, \alpha_i \rangle e_i \\ [h, h'] &= 0 & [h, f_i] &= -\langle h, \alpha_i \rangle f_i. \end{aligned}$$

Let $\mathfrak{m}$ be the unique maximal ideal which intersects $\mathfrak{h}$ trivially. Then the Kac-Moody algebra with Cartan matrix A is defined to be

$$\mathfrak{g}(A) := \bar{\mathfrak{g}}(A)/\mathfrak{m}.$$

One can check that the generators of $\mathfrak{g}(A)$ satisfy the Serre's relations:

$$(\text{ad } e_i)^{1-a_{ij}} e_j = 0$$
$$(\text{ad } f_i)^{1-a_{ij}} f_j = 0$$

If the matrix A is "symmetrizable" then these relations generate the ideal $\mathfrak{m}$. In other words, these are the only additional relations needed to define $\mathfrak{g}(A)$.

A matrix A is called *symmetrizable* if there exists an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$. In this case, $\mathfrak{g}(A)$ is also called symmetrizable. The Kac-Moody algebra $\mathfrak{g}(A)$ is symmetrizable if and only if there exists a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}(A)$.

Affine Lie algebras are Kac-Moody algebras. If A is the Cartan matrix of $\widehat{\mathfrak{g}}$, then there is an isomorphism $\widehat{\mathfrak{g}} \cong \mathfrak{g}(A)$ which can be defined using the Chevalley generators. Similarly, finite-dimensional simple Lie algebras and twisted affine Lie algebras are Kac-Moody algebras.

Given a Kac-Moody algebra $\mathfrak{g}(A)$ such that A is the Cartan matrix on an affine Lie algebra, we can recover the elements K and δ which were previously defined. It follows from $[K, e_i] = 0$ for $i = 0, 1, \ldots, n$ that $\langle K, \alpha_i \rangle = 0$ for $i = 0, 1, \ldots, n$. Let $[a_i^\vee]$ be a row vector such that $[a_i^\vee]A = 0$. It is determined up to a scalar since the corank of A is 1. Then

$$K = \sum_{i=0}^{n} a_i^\vee \alpha_i^\vee.$$

Similarly, $\langle \alpha_i, \delta \rangle = 0$ for $i = 0, 1, \ldots, n$. So if $[a_i]$ is a column vector such that $A[a_i] = 0$, then

$$\delta = \sum_{i=0}^{n} a_i \alpha_i.$$

Simple finite-dimensional, affine and twisted affine Lie algebras are all the Kac-Moody algebras of "finite growth". The *principal grading*

$$\mathfrak{g} = \oplus_{m \in \mathbb{Z}} \mathfrak{g}(m)$$

of $\mathfrak{g}$ for a set of simple roots Π is defined by $\mathfrak{g}(0) = \mathfrak{h}$ and

$$\mathfrak{g}(1) = \oplus_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}.$$

Then $\mathfrak{g}$ is said to have *finite growth* if dim $\mathfrak{g}(m)$ grows polynomially with respect to m, and the Gelfand-Kirillov dimension is the degree of this polynomial. For affine and twisted affine Lie algebras, the Gelfand-Kirillov dimension is 1.

4. Affine Weyl group

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and fix a base $\Pi \in \Delta$. Let $(\cdot, \cdot)$ be the non-degenerate invariant symmetric bilinear form normalized such that $(\theta, \theta) = 2$. Recall that θ is the highest root of Δ_+.

For $j = 0, 1, \ldots, n$, define $\Lambda_j \in \widehat{\mathfrak{h}}^*$ by

$$\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij} \text{ for } i = 0, \ldots, n, \text{ and } \langle d, \lambda_j \rangle = 0.$$

We identify $\widehat{\mathfrak{h}}$ with $\widehat{\mathfrak{h}}^*$ via the linear map $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ defined by $\nu(\alpha_i^\vee) = \frac{(\alpha_i^\vee, \alpha_i^\vee)}{2}\alpha_i$, $\nu(K) = \delta$, $\nu(d) = a_0\Lambda_0$. For $\alpha, \beta \in \widehat{\mathfrak{h}}^*$, define $(\alpha, \beta) := (\nu^{-1}(\alpha), \nu^{-1}(\beta))$.

Simple reflections at the roots $\alpha_i : i = 0, 1 \ldots, n$ are defined as follows. For $\lambda \in \widehat{\mathfrak{h}}^*$, let

$$r_i(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i.$$

The affine Weyl group $\widehat{W}$ is the group generated by $r_0, r_1, \ldots, r_n$. The subgroup W generated by $r_1, \ldots, r_n$ is canonically isomorphic to the Weyl group of $\mathfrak{g}$.

To simplify notation, fix $\lambda \in \widehat{\mathfrak{h}}^*$ and set $k = \langle \lambda, K \rangle = (\lambda, \delta)$. Then

$$r_0(\lambda) = \lambda - ((\lambda, \theta) - k)(\delta - \theta)$$

and

$$r_0 r_\theta(\lambda) = \lambda - ((\lambda, \theta) + k)\delta + k\theta.$$

For $\mu \in \mathfrak{h}^*$ we define the translation

$$t_\mu(\lambda) = \lambda + k\mu - ((\lambda, \mu) + k\frac{(\mu, \mu)}{2})\delta.$$

One can check that

$$\begin{aligned} t_\theta &= r_0 r_\theta \\ t_\mu t_{\mu'} &= t_{\mu+\mu'} \\ t_{w\mu} &= w t_\mu w^{-1} \qquad \text{for all } w \in W. \end{aligned}$$

Hence,

$$T := \{t_\mu \mid \mu \in \mathbb{Z}(W\theta)\}$$

is a normal subgroup of $\widehat{W}$. Since W is finite and T is free abelian, $W \cap T = 1$. Thus, $W \ltimes T$ is a subgroup of $\widehat{W}$. Now $r_i \in W \subset \widehat{W}$ for $i = 1, \ldots, n$ and $r_0 = t_\theta r_\theta \in W \ltimes T$. Since $r_0, r_1, \ldots, r_n$ generate $\widehat{W}$, we conclude that

$$\widehat{W} = W \ltimes T.$$