Introduction to affine Lie algebras
Notes for Lecture 3
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1. Kac-Moody algebra

An $n \times n$-matrix is called a *generalized Cartan matrix* if for $i, j = 1, \ldots, n$,

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

Let A be a generalized Cartan matrix, and let $\mathfrak{h}$ be a vector space (over $\mathbb{C}$) with dimension $n + \text{corank}(A)$. Let

$$\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$$
$$\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$$

be linearly independent sets satisfying

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}.$$

Define $\bar{\mathfrak{g}}(A)$ by generators: $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, $\mathfrak{h}$ and relations:

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee \qquad [h, e_i] = \langle h, \alpha_i \rangle e_i$$
$$[h, h'] = 0 \qquad [h, f_i] = -\langle h, \alpha_i \rangle f_i.$$

Let $\mathfrak{m}$ be the unique maximal ideal which intersects $\mathfrak{h}$ trivially. Then the Kac-Moody algebra with Cartan matrix A is defined to be

$$\mathfrak{g}(A) := \bar{\mathfrak{g}}(A)/\mathfrak{m}.$$

2. Integrable modules

For each $i = 1, \ldots, n$, let $\mathfrak{s}_i$ be the Lie subalgebra with vector space basis $\{e_i, f_i, \alpha_i^\vee\}$. Then $\mathfrak{s}_i$ is isomorphic to $\mathfrak{sl}_2$, and $\mathfrak{s}_i$ acts on $\mathfrak{g}(A)$ via the adjoint action.

Lemma 2.1. *The generators $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ satisfy the Serre's relations ($i \neq j$):*

$$(ad\ e_i)^{1-a_{ij}} e_j = 0$$
$$(ad\ f_i)^{1-a_{ij}} f_j = 0.$$

Proof. We prove the second relation. The proof of the first is similar. Fix $i, j \in \{1, \ldots, n\}$: $i \neq j$. Claim: $(\text{ad } e_k)((\text{ad } f_i)^{1-a_{ij}} f_j) = 0$ for $k = 1, \ldots, n$.

If $k \neq i$, then $[e_k, f_i] = 0$ implies that

$$\begin{aligned}(\text{ad } e_k)(\text{ad } f_i)^{1-a_{ij}} f_j &= (\text{ad } f_i)^{-a_{ij}}[f_i, [e_k, f_j]] \\ &= \delta_{kj}(\text{ad } f_i)^{-a_{ij}}[f_i, h_j] \\ &= \delta_{kj} a_{ij}(\text{ad } f_i)^{-a_{ij}} f_i \\ &= 0\end{aligned}$$

If $k = i$, then

$$V_{ij} := \sum_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}((\text{ad } f_i)^m f_j)$$

is an $\mathfrak{s}_i$-module with highest vector f_j and highest weight $-a_{ij}$. Indeed, $[e_i, f_j] = 0$, $[h_i, f_j] = -a_{ij} f_j$ and $U(\mathfrak{s}_i)v_j = V_{ij}$. Hence, $(\text{ad } e_i)((\text{ad } f_i)^{1-a_{ij}} f_j) = 0$ by standard $\mathfrak{sl}_2$-theory.

Let $w_{ij} = (\text{ad } f_i)^{1-a_{ij}} f_j$. Then we have shown that $(\text{ad } e_k) w_{ij} = 0$ for $k = 1, \dots, n$. So $I := U(\mathfrak{n}_-)U(\mathfrak{h}) w_{ij}$ is an ideal in $\mathfrak{g}(A)$ which is contained in $\mathfrak{n}_-$. Since $I \cap \mathfrak{h} = 0$, we conclude that $I \subset \mathfrak{m}$ and in particular that $w_{ij} = 0$. □

Definition 2.2. Let V be a module for a Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is *locally nilpotent* on V if for any $v \in V$ there exists $m \in \mathbb{Z}_+$ such that $x^m v = 0$.

Lemma 2.3. *If $\{y_i\}_{i \in I} \subset \mathfrak{g}$ generate $\mathfrak{g}$ (as a Lie algebra) and $x \in \mathfrak{g}$ such that for each $i \in I$ there exists $N_i \in \mathbb{Z}_+$ so that $(ad\ x)^{N_i} y_i = 0$, then ad x is locally nilpotent on $\mathfrak{g}$.*

Proof. Use Leibnitz rule $(\text{ad } x)^k [y, z] = \sum_{i=0}^{k} \binom{k}{i} \left[(\text{ad } x)^i y, (\text{ad } x)^{k-i} z\right]$ and induction. □

Lemma 2.4. *ad e_i and ad f_i are locally nilpotent on $\mathfrak{g}(A)$*

Proof. For each $i = 1, \dots, n$, the defining relations of $\mathfrak{g}(A)$ imply that

$$(\text{ad } e_i)^2 h = (\text{ad } f_i)^2 h = 0$$

for all $h \in \mathfrak{h}$. Also,

$$(\text{ad } e_i)^3 f_j = (\text{ad } f_i)^3 e_j = 0$$

for $j = 1, \dots, n$, and $[e_i, e_i] = 0$. We see that by taking $x = e_i$ or $x = f_i$, the set $\{e_1, \dots, e_n, f_1, \dots, f_n, \} \cup \mathfrak{h}$ satisfies the hypothesis of the previous lemma. Therefore, ad e_i and ad f_i are locally nilpotent on $\mathfrak{g}(A)$. □

Definition 2.5. A $\mathfrak{g}(A)$-module V is called a *weight module* if

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu,$$

where $V_\mu = \{v \in V \mid hv = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}$. If $V_\mu \neq 0$, then μ is called a weight.

Definition 2.6. A weight module V is called *integrable* if $e_1, \dots, e_n, f_1, \dots, f_n$ are locally nilpotent on V.

The adjoint module of $\mathfrak{g}(A)$ is integrable by Lemma 2.4.

Proposition 2.7. *If V is an integrable $\mathfrak{g}(A)$-module, then for each $i = 1, \dots, n$, V decomposes into a direct sum of finite-dimensional irreducible $\mathfrak{h}$-invariant $\mathfrak{s}_i$-modules.*

Proof. For each $v \in V$, $v = \sum_{\mu \in X} k_\mu v_\mu$ where X is a finite subset of $\mathfrak{h}^*$. The subspace

$$U = \sum_{k,m \in \mathbb{Z}_{\geq 0}} \sum_{\mu \in X} \mathbb{C} f_i^k e_i^m (v_\mu)$$

is an $\mathfrak{h}$-invariant $\mathfrak{s}_i$-module. Also, U is finite dimensional, since e_i and f_i are locally nilpotent on V. So by Weyl's theorem, U decomposes into a direct sum of irreducible $\mathfrak{h}$-invariant s_i-modules. Hence, each $v \in V$ lies in a finite sum of finite-dimensional irreducible $\mathfrak{h}$-invariant s_i-submodules.

Using Zorn's Lemma, one can prove that there exists a maximal $\mathfrak{h}$-invariant completely reducible $\mathfrak{s}_i$-module $V' \subset V$. Suppose that $V' \neq V$, and let $x \in V \setminus V'$. Then by the previous paragraph there exist finite-dimensional $\mathfrak{h}$-invariant irreducible s_i-submodules U_j, $j = 1, \dots, N$, such that $x \in \oplus_{j=1}^N U_j \subset V$. Since $x \notin V'$, there exists U_k such that $U_k \not\subset V'$. Since U_k is an irreducible module, this implies that $U_k \cap V' = 0$. Then $U_k \oplus V'$ is an $\mathfrak{h}$-invariant completely reducible s_i-module, which contradicts the maximality of V'. ☐

3. The Weyl group of $\mathfrak{g}(A)$

The Weyl group of $\mathfrak{g}(A)$ is defined as follows. For each $i = 1, \dots, n$ and $\lambda \in \mathfrak{h}^*$ define

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i.$$

Then $r_i(\alpha_i) = -\alpha_i$ since $a_i i = 2$, and $H_i = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$ is fixed. So r_i defines a reflection on $\mathfrak{h}^*$. The Weyl group W for $\mathfrak{g}(A)$ is the group generated by $r_1, \dots, r_n$.