Introduction to affine Lie algebras
Notes for Lecture 4
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1. Endomorphisms of V

Let $\mathfrak{g}(A)$ be a Kac-Moody algebra and let V be a $\mathfrak{g}(A)$-module. Let $x \in \mathfrak{g}(A)$ such that x is locally nilpotent on V. Let

$$\exp\, x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Then for each $v \in V$ there exist $N \in \mathbb{Z}+$ such that

$$(\exp\, x)v = \sum_{n=0}^{N} \frac{1}{n!} x^n v,$$

so exp x is a well-defined endomorphism of V.

If V is an integrable module, then for $i = 1, \dots, n$, exp e_i and exp f_i are well-defined endomorphisms of V. Consider the adjoint representation of $\mathfrak{g}(A)$.

Lemma 1.1. *exp(ad e_i) and exp(ad f_i) are automorphisms of $\mathfrak{g}(A)$.*

This follows from the next claim applied to the adjoint representation.

Claim 1.2. *Let $x, y \in End(V)$ such that x is locally nilpotent on V and there exists $m \in \mathbb{Z}_+$ such that $x^m y = 0$, then*

$$(exp(ad\ x))y = (exp\ x)y(exp\ (-x)) = (exp\ x)y(exp\ x)^{-1}.$$

Proof. For $v \in V$ there exist $n \in \mathbb{Z}_+$ such that $x^n v = 0$. Let $M = \max\{m, n\}$. Then

$$((\exp(\text{ad } x))y)v = (\sum_{k=0}^{M} \frac{1}{k!} (\text{ad } x)^k y)v = \left(\sum_{k=0}^{M} \frac{1}{k!} \sum_{s=0}^{k} (-1)^s \binom{k}{s} x^{k-s} y x^s \right) v$$
$$= \left((\sum_{s=0}^{M} \frac{1}{s!} x^s) y (\sum_{t=0}^{M} \frac{1}{t!} (-x)^t) \right) v = ((\exp\, x) y (\exp\, (-x)))v.$$

□

2. The Weyl group of $\mathfrak{g}(A)$

The Weyl group of $\mathfrak{g}(A)$ is defined as follows. For each $i = 1, \dots, n$ and $\lambda \in \mathfrak{h}^*$ define

$$r_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i.$$

Then $r_i(\alpha_i) = -\alpha_i$ since $a_{ii} = 2$, and $H_i = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}$ is fixed. So r_i defines a reflection on $\mathfrak{h}^*$. The Weyl group W for $\mathfrak{g}(A)$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by $r_1, \dots, r_n$.

3. Integrable modules

Let V be an integrable $\mathfrak{g}(A)$-module. Then $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$. Define

$$R_i = (\exp\, f_i)(\exp\, (-e_i))(\exp\, f_i).$$

Lemma 3.1. *For $i = 1, \ldots, n$ and $\lambda \in \mathfrak{h}^*$,*

$$R_i(V_\lambda) = V_{r_i(\lambda)}.$$

If V is the adjoint module, then R_i is an automorphism of $\mathfrak{g}(A)$ which we denote by R_i^{ad} and the restriction to $\mathfrak{h}$ is given by

$$R_i^{ad}|_{\mathfrak{h}} = r_i.$$

It follows that the set of weights for V is invariant under the action of W. Moreover, $\dim\, V_\lambda = \dim\, V_{w(\lambda)}$ for all $\lambda \in \mathfrak{h}^*$ and $w \in W$.

Example 3.2. $\mathfrak{sl}(2)$ has a basis $\{e, f, h\}$ which satisfies: $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Consider the standard representation of $\mathfrak{sl}(2)$. It is a two dimension module with weight space decomposition $V = V_1 \oplus V_{-1}$. The action on V can be represented by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $V_1 = \mathbb{C}E_1$ and $V_{-1} = \mathbb{C}E_2$. One can check that

$$\exp\, f_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \exp\, (-e_i) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

and

$$R_i = (\exp\, f_i)(\exp\, (-e_i))(\exp\, f_i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see that $R_i(E_1) = E_2$ and $R_i(E_2) = -E_1$. Moreover, $(R_i)^2(E_1) = -E_1$, $(R_i)^2(E_2) = -E_2$.

In general, if V is an integrable module and $v \in V_\lambda$ then

$$(R_i)^2(v) = (-1)^{\langle \alpha_i^\vee, \lambda\rangle} v.$$

We note that $\langle \alpha_i^\vee, \lambda\rangle \in \mathbb{Z}$ since V decomposes into direct sum of finite-dimensional simple $\mathfrak{s}_i = \mathrm{span}\{e_i, f_i, \alpha_i^\vee\} \cong \mathfrak{sl}(2)$ modules.

4. Highest weight modules

If $\mathfrak{g}$ is a Lie algebra, then the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ can be realized as the quotient of the tensor algebra $T(\mathfrak{g})$ by the two sided ideal J generated by all elements $x \otimes y - y \otimes x - [xy]$ with $x, y \in \mathfrak{g}$.

We have a triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

which induces a decomposition

$$U(\mathfrak{g}(A)) = U(\mathfrak{n}_-)U(\mathfrak{h})U(\mathfrak{n}_+).$$

Definition 4.1. A $\mathfrak{g}(A)$-module V is called a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $v \in V$ such that

$$n_+ v = 0, \qquad hv = \langle h, \lambda \rangle v \text{ for all } h \in \mathfrak{h}, \qquad U(\mathfrak{g}(A))v = V.$$

In this case, $U(\mathfrak{n})v = V$ and

$$V = \oplus_{\mu \leq \lambda} V_\mu.$$

Recall that $\mu \leq \gamma$ is a partial order defined on $\mathfrak{h}^*$ by $\mu \leq \gamma$ if and only if

$$\gamma - \mu \in Q^+ := \sum_{\alpha_i \in \Pi} \mathbb{Z}_{\geq 0} \alpha_i.$$

Definition 4.2. The Verma module with highest weight $\Lambda \in \mathfrak{h}^*$ is defined as follows. Let $\mathbb{C}_\Lambda$ be a one-dimensional $U(\mathfrak{n}_+ \oplus \mathfrak{h})$ module with the action on $1 \in \mathbb{C}$ defined by

$$n_+(1) = 0, \qquad h(1) = \langle h, \Lambda \rangle 1 \text{ for all } h \in \mathfrak{h}.$$

Then

$$M(\Lambda) = U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{n}_+ \oplus \mathfrak{h})} \mathbb{C}_\Lambda$$

is a highest weight module called a *Verma module* with highest weight Λ.

$M(\Lambda)$ has a unique maximal proper submodule, since the sum of any two proper submodules is a proper submodule (it does not contain the highest weight vector v_Λ). Let $L(\Lambda)$ be the unique simple quotient of $M(\Lambda)$.

Let $P(\Lambda)$ denote the set of weights of $L(\Lambda)$. Since $M(\lambda)$ is a highest weight module, $P(\Lambda) \subset \lambda - Q^+$.

Lemma 4.3. *$L(\Lambda)$ is integrable if and only if*

$$\Lambda \in P_+ := \{\mu \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \mu \rangle \in \mathbb{Z}_{\geq 0}, \ i = 1, \ldots, n\}.$$

Hence, if $\Lambda \in P_+$, then $P(\Lambda)$ is W-invariant and $\mathrm{mult}(\gamma) = \mathrm{mult}(w(\gamma))$ for all $\gamma \in P(\Lambda)$ and all $w \in W$.

5. Real and Imaginary roots

Let $\mathfrak{g}(A)$ be a Kac-Moody algebra. A root $\beta \in \Delta$ is called *real* if there exists $w \in W$ such that $w(\beta)$ is a simple root, that is, $w(\beta) \in \Pi$. Otherwise it is called *imaginary*. Then $\Delta = \Delta^{re} \sqcup \Delta^{im}$.

If $\alpha \in \Delta$ is real then $\dim \mathfrak{g}(A)_\alpha = 1$.

If $\widehat{\mathfrak{g}}$ is an affine Lie algebra, then

$$\widehat{\Delta}^{re} = \{m\delta + \alpha\}_{m \in \mathbb{Z}, \alpha \in \Delta}, \qquad \widehat{\Delta}^{im} = \{m\delta\}_{m \in \mathbb{Z} \setminus \{0\}}.$$

Note that $w(\delta) = \delta$ for all $w \in W$.