LIE ALGEBRAS: HOMEWORK 1
DUE: 23 MARCH 2010

(1) Let $A = k[t]$ be the algebra of polynomials, and fix $f \in A$.
 (a) Define $d : A \to A$ by the formula
$$d(g) = fg'.$$
 Prove that d is a derivation.
 (b) Define $d : A \to A$ by the formula
$$d(g) = fg' + g.$$
 Is this a derivation?
 (c) Prove that any derivation is of the form described in (a). (Consider the value of d on $1, t \in A$.)

(2) Let A be an algebra. Prove that $\mathrm{Der}(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

(3) Prove that $\mathfrak{sl}_n$ is an ideal in $\mathfrak{gl}_n$. What is the quotient algebra? Present $\mathfrak{gl}_n$ as a direct product of two algebras.

(4) Let E_{ij} be the standard basis of $\mathfrak{gl}_n$, (the matrix E_{ij} has a 1 in the ij-coordinate and zeros everywhere else). Write the structure constants of $\mathfrak{gl}_n$ wrt this basis (i.e. express the brackets $[E_{ij}, E_{kl}]$ as a linear combination of basis elements.

(5) Prove that the following are Lie subalgebras of $\mathfrak{gl}_n$:
 (a) $\mathfrak{b}_n = \{a \in \mathfrak{gl}_n \mid a_{ij} = 0 \text{ for } i > j\}$ - upper triangular matrices;
 (b) $\mathfrak{n}_n = \{a \in \mathfrak{gl}_n \mid a_{ij} = 0 \text{ for } i \geq j\}$ - strictly upper triangular matrices;
 (c) $D = \{a \in \mathfrak{gl}_n \mid a_{ij} = 0 \text{ for } i \neq j\}$ - diagonal matrices.

16 March 2010