LIE ALGEBRAS: HOMEWORK 10
DUE: 15 JUNE 2010

Let E be a Euclidean space, and Δ a root system in E. Let W be the Weyl group of Δ.

(1) Define $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. Call $\Delta^\vee := \{\alpha^\vee \mid \alpha \in \Delta\}$ the dual of Δ. Prove that $\Delta^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to the Weyl group W of Δ.

(2) Let Δ be irreducible. Prove that $\Delta^\vee$ is irreducible. Prove that if Δ has all roots of equal length, then so does $\Delta^\vee$ and in this case $\Delta^\vee$ is isomorphic to Δ. Otherwise, if Δ has two root lengths, then so does $\Delta^\vee$, such that if α is long then $\alpha^\vee$ is short (and vice versa). Hence, each irreducible root system is isomorphic to its dual except, B_l and C_l which are dual to each other.

(3) Let Π be a base for Δ. Let λ be in the fundamental Weyl chamber relative to Π, $\lambda \in \mathcal{C}(\Pi) := \{v \in E \mid (v,\alpha) > 0 \text{ for all } \alpha \in \Pi\}$. Prove that if $\sigma(\lambda) = \lambda$ for some $\sigma \in W$, then $\sigma = 1$.

(4) Prove that the only reflections in W are those of the form σ_α with $\alpha \in \Delta$. (Hint: A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in W.

(5) Prove that the Weyl group of a root system Δ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.

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