LIE ALGEBRAS: HOMEWORK 11
DUE: 22 JUNE 2010

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. For $x \in \mathfrak{g}$, extend ad x to an endomorphism of $\mathcal{U}(\mathfrak{g})$ by defining ad $x(y) = xy - yx$ for all $y \in \mathcal{U}(\mathfrak{g})$.

(1) Prove that for $x,\ x_1, \ldots, x_m \in \mathfrak{g}$, that

$$\text{ad } x\ (x_1 \cdots x_m) = \sum_{i=0}^{m} x_1 x_2 \cdots \text{ad } x(x_i) \cdots x_m.$$

(Note that $x_1 \cdots x_m \in \mathcal{U}(\mathfrak{g})$.)

(2) Let $\mathcal{U}(\mathfrak{g})_n$ be the n^{th} filtration of $\mathcal{U}(\mathfrak{g})$. Prove that

$$[\mathcal{U}(\mathfrak{g})_n, \mathcal{U}(\mathfrak{g})_m] \subset \mathcal{U}(\mathfrak{g})_{n+m-1}.$$

(Hint: Use problem 1.)

(3) Prove that each element of $\mathcal{U}(\mathfrak{g})$ lies in a finite dimensional $\mathfrak{g}$-submodule. (Hint: Use problem 2.)

(4) Prove that $\mathcal{U}(\mathfrak{g})$ has no zero divisors. (Hint: Use the fact that the associated graded algebra $\mathrm{Gr}(\mathcal{U}(\mathfrak{g}))$ is isomorphic to a polynomial algebra.)

15 June 2010