LIE ALGEBRAS: HOMEWORK 13
DUE: 6 JULY 2010

Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Let Π be a base for the corresponding root system Δ.

(1) Let $\lambda \in \mathfrak{h}^*$. Prove that the left ideal $I(\lambda)$ of $\mathcal{U}(\mathfrak{g})$, (which is generated by $\mathfrak{n}^+$ and by the elements of the form $h - \lambda(h)1$ with $h \in \mathfrak{h}$), is already generated by the elements $x_\alpha \in \mathfrak{g}_\alpha$, $h_\alpha - \lambda(h_\alpha)1$ with $\alpha \in \Pi$.

(2) For $\mu \in \mathfrak{h}^*$, define $K(\mu)$ to be the number of distinct sets of non-negative integers $\{k_\alpha\}_{\alpha\in\Delta^+}$ for which $\mu = \sum_{\alpha\in\Delta^+} k_\alpha \alpha$. The function $K(\mu)$ is called the *Kostant partition function*. Let $M(\lambda)$ be the Verma module with highest weight λ. Prove that $\dim M(\lambda)_\mu = K(\lambda - \mu)$ by describing a basis for the weight space $M(\lambda)_\mu$.

(3) For a weight module V, let $X(V)$ denote the set of weights of V. Let V and W be finite dimensional $\mathfrak{g}$-modules. Prove that

$$X(V \otimes W) = \{\nu + \nu' \mid \nu \in X(V),\ \nu' \in X(W)\}$$

and that $\dim(V \otimes W)_{\nu+\nu'}$ equals

$$\sum_{\substack{\beta \in X(V),\ \beta' \in X(W) \\ \beta + \beta' = \nu + \nu'}} \dim V_\beta \cdot \dim W_{\beta'}.$$

(4) Let $P^+(\Pi) = \{\lambda \in \mathfrak{h}^* \mid < \lambda, \alpha > \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi\}$, and let $\lambda \in P^+(\Pi)$. Prove that 0 occurs as a weight of $M(\lambda)$ if and only if λ is a sum of roots.

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