LIE ALGEBRAS: HOMEWORK 2
DUE: 6 APRIL 2010

(1) Let
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
be an ordered basis for $\mathfrak{sl}_2(k)$. Compute the matrices of $\operatorname{ad} e$, $\operatorname{ad} h$, $\operatorname{ad} f$ relative to this basis.

(2) Let L be a Lie algebra. Prove that the set of all inner derivations $ad\ L = \{\operatorname{ad} x \mid x \in L\}$ is an ideal in $\operatorname{Der} L$. What can we say about L if $ad\ L = \{0\}$?

(3) Let $L = \mathfrak{gl}_n$, and prove that $[L, L] = \mathfrak{sl}_n$. If $L = \mathfrak{sl}_n$, then what is $[L, L]$?

(4) Let L be a Lie algebra, and let K be an ideal of L such that L/K is nilpotent and $\operatorname{ad} x|_K$ ($\operatorname{ad} x$ restricted to the ideal K) is nilpotent for all $x \in L$. Prove that L is nilpotent.

(5) (a) Let $M \in \mathfrak{gl}_m\mathbb{C}$. Define
$$L_M = \{X \in \mathfrak{gl}_m\mathbb{C} \mid X^t M + MX = 0\}.$$
Prove that L_M is a Lie subalgebra of $\mathfrak{gl}_m\mathbb{C}$.

(b) **The symplectic Lie algebra** $\mathfrak{sp}_{2n}\mathbb{C}$ is defined by taking
$$M := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathfrak{gl}_{2n}\mathbb{C}.$$
Show that
$$\mathfrak{sp}_{2n}\mathbb{C} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}_{2n} \mid A, B, C, D \in \mathfrak{gl}_n,\ B = B^t,\ C = C^t,\ A = -D^t \right\}.$$

Note . We can define the **orthogonal Lie algebra** by the same method. For $\mathfrak{o}_{2n}\mathbb{C}$ we take $M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, and for $\mathfrak{o}_{2n+1}\mathbb{C}$ we take $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$. See Humphreys Section 1.2 for details.

23 March 2010