LIE ALGEBRAS: HOMEWORK 4
DUE: 27 APRIL 2010

Assume that $\mathbb{F}$ is an algebraically closed field of characteristic zero and that L is a finite dimensional Lie algebra over $\mathbb{F}$.

(1) Let L be a nilpotent Lie algebra. Prove that the Killing form of L is identically zero.

(2) Prove that a Lie algebra L is solvable if and only if $[L, L]$ lies in the the kernel of the Killing form.

(3) Let L be the two dimensional non-abelian Lie algebra. Show that L is solvable. Prove that the Killing form of L is non-trivial.

(4) Let $L = \mathfrak{sl}_2(\mathbb{F})$. Compute the basis of L dual to the standard basis $\{e, h, f\}$ relative to the Killing form.

(5) Let $L = \mathfrak{sl}_n(\mathbb{F})$, and let $g \in GL_n(\mathbb{F})$. Define a map

$$\begin{aligned} \phi_g : &L \longrightarrow L \\ &x \mapsto -gx^T g^{-1}, \end{aligned}$$

where x^T is the transpose of x. Prove that ϕ_g is a Lie algebra automorphism of $L = \mathfrak{sl}_n(\mathbb{F})$. First show that this map well-defined (i.e $\phi_g(L) \subseteq L$).

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