LIE ALGEBRAS: HOMEWORK 5
DUE: 4 MAY 2010

Assume that $\mathbb{F}$ is an algebraically closed field with characteristic zero, and that L is a finite dimensional Lie algebra.

(1) (a.) Let $L = \mathfrak{sl}_2(\mathbb{F})$. Using the standard basis $\{e, h, f\}$ for L, write down the Casimir element of the adjoint representation.

(b.) Let $L = \mathfrak{sl}_3(\mathbb{F})$. Using the standard basis $\{E_{ij}, H_i := E_{ii} - E_{i+1,i+1}\}$ for L, write down the Casimir element of the natural representation, by first computing the dual basis relative to the trace form.

(2) Let V be a finite dimensional L-module. Prove that V is a direct sum of irreducible L-submodules if and only if for each L-submodule of V there exists a complement submodule.

(3) Let L be a solvable Lie algebra, and let V be an irreducible finite dimensional L-module. Prove that the Lie subalgebra $[L, L]$ acts trivially on V.

(4) Let V and W be L-modules, and let $V \otimes W$ be the tensor product of the underlying vector spaces. Define an action of L on $V \otimes W$ by

$$x.(v \otimes w) := (x.v) \otimes w \; + \; v \otimes x.w$$

where $x \in L$, $v \in V$, $w \in W$ (and extend linearly). Verify that this defines an L-module structure on $V \otimes W$.

Definition of tensor product of vector spaces: Let V and W be vector spaces over $\mathbb{F}$. Then the tensor product of V and W over $\mathbb{F}$ is the Cartesian product $V \times W$, modulo the subspace generated by the following relations:

$$\begin{aligned} (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2 \\ av \otimes w \;=\; v \otimes aw &= a(v \otimes w), \end{aligned}$$

where $a \in \mathbb{F}$, $v \in V$, $w \in W$. If $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W, then $\{v_i \otimes w_j\}$ is a basis for the vector space $V \otimes W$.

Extra Credit (+10): Let L be a simple Lie algebra. Suppose that $\alpha(x, y)$ and $\beta(x, y)$ are non-degenerate symmetric invariant bilinear forms on L. Prove that $\alpha(x, y)$ and $\beta(x, y)$ are proportional.

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