LIE ALGEBRAS: HOMEWORK 6
DUE: 11 MAY 2010

Assume that $\mathbb{F}$ is an algebraically closed field with characteristic zero, and that L is a finite dimensional Lie algebra.

(1) Let triv be the one dimensional module on which L acts trivially (by zero). Let V be an L-module. The dual module V^* is defined to be the L-module $V^* := \operatorname{Hom}(V, \mathsf{triv})$. Explicitly, the action of L is given by

$$(x.f)(v) := -f(x.v)$$

where $x \in L, f \in V^* = \operatorname{Hom}(V, \mathsf{triv}), v \in V$. Let V and W be finite dimensional L-modules, and let $\{f_1, \ldots, f_n\}$ be a basis for V^* and $\{w_1, \ldots, w_m\}$ be a basis for W. Prove that the map defined (on the basis) by:

$$V^* \otimes W \to \operatorname{Hom}(V, W)$$
$$f_i \otimes w_j \mapsto \phi_{f_i \otimes w_j}(v) := f_i(v) w_j,$$

defines an L-module isomorphism, thus showing that $V^* \otimes W \cong \operatorname{Hom}(V, W)$ as L-modules.

(2) Decompose the tensor product of the two $\mathfrak{sl}_2(\mathbb{C})$-modules $V(3)$, $V(5)$ into a direct sum of irreducible submodules (abstractly). What is the general formula for the decomposition of $V(m) \otimes V(n)$ into a direct sum of irreducible submodules?

(3) A Lie algebra L is called reductive if Rad $L = Z(L)$. Prove that if L is reductive, then L is a completely reducible (ad L)-module. Then show that L is the direct sum of $Z(L)$ and $[L, L]$, with $[L, L]$ semisimple.

(4) Prove that if L is a completely reducible (ad L)-module, then L is reductive.

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