LIE ALGEBRAS: HOMEWORK 7
DUE: 25 MAY 2010

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Let D be the set of diagonal matrices in $\mathfrak{gl}_n(\mathbb{F})$. Then $\{E_{11}, \ldots, E_{nn}\}$ is a basis for the vector space D. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the dual basis for D^* defined by $\varepsilon_i(E_{jj}) = \delta_{ij}$. Then if $d \in D$ with $d = \sum_{i=1}^n d_i E_{ii}$ $(d_i \in \mathbb{F})$, then $\varepsilon_i(d) = d_i$. Let $M(k, j)$ denote the set of $k \times j$-matrices over $\mathbb{F}$.

There are four families of classical Lie algebras:

(1) $A_r := \mathfrak{sl}_{r+1}(\mathbb{F}) = \{x \in \mathfrak{gl}_{r+1}(\mathbb{F}) \mid \mathrm{Tr}(x) = 0\}$, the special linear algebra;

(2) $$B_r := \mathfrak{o}_{2r+1}(\mathbb{F}) = \left\{ \begin{pmatrix} 0 & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix} \in \mathfrak{gl}_{2r+1}(\mathbb{F}) \;\middle|\; \begin{array}{l} b_1, b_2 \in M(1,r),\ c_1, c_2 \in M(r,1), \\ m,n,p,q \in M(r,r) \\ c_1 = -b_2^t,\ c_2 = -b_1^t, \\ n = -n^t,\ p = -p^t,\ q = -m^t \end{array} \right\},$$
the orthogonal algebra;

(3) $$C_r := \mathfrak{sp}_{2r}(\mathbb{F}) = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathfrak{gl}_{2r}(\mathbb{F}) \;\middle|\; \begin{array}{l} m,n,p,q \in M(r,r), \\ n = n^t,\ p = p^t,\ q = -m^t \end{array} \right\},$$
the symplectic algebra;

(4) $$D_r := \mathfrak{o}_{2r}(\mathbb{F}) = \left\{ \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathfrak{gl}_{2r}(\mathbb{F}) \;\middle|\; \begin{array}{l} m,n,p,q \in M(r,r), \\ n = -n^t,\ p = -p^t,\ q = -m^t \end{array} \right\},$$
the orthogonal algebra.

Exercises (1)-(4): For each classical Lie algebra $\mathfrak{g}$, prove that $\mathfrak{h} := D \cap \mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ consisting of ad-semisimple elements, with dimension r. Determine the root spaces of $\mathfrak{g}$ with respect to this Cartan subalgebra. Determine the roots, expressed in terms of the $\varepsilon_1, \ldots, \varepsilon_n$ defined above.

Exercise (5): Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$. By Lemma 2.6, for each $\alpha \in \Delta$ and nonzero $x_\alpha \in \mathfrak{g}_\alpha$ there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\{x_\alpha,\ y_\alpha,\ h_\alpha := [x_\alpha, y_\alpha]\}$ span a three dimensional subalgebra isomorphic to $\mathfrak{sl}_2$. In particular, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one dimensional subspace of $\mathfrak{h}$. For each $\alpha \in \Delta$, what is this subspace?

Remark 0.1. We showed in class that $\mathfrak{sl}_n(\mathbb{F})$ is simple (hence, semisimple), and so $\mathfrak{h}$ as defined above is actually a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{F})$. We will see that the other classical algebras are also simple.

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