LIE ALGEBRAS: HOMEWORK 8
DUE: 1 JUNE 2010

Let E be a Euclidean space, and let Δ be a root system in E with Weyl group W.

(1) Let E' be a subspace of E. If a reflection σ_λ ($\lambda \in E$) leave E' invariant, prove that either $\lambda \in E'$ or $E' \subset P_\lambda$.

(2) In Table 1, show that the order of $\sigma_\alpha \sigma_\beta$ in W is (respectively) $2,\ 3,\ 4,\ 6$ when $\theta = \frac{\Pi}{2}$, $\frac{\Pi}{3}$ (or $\frac{2\Pi}{3}$), $\frac{\Pi}{4}$ (or $\frac{3\Pi}{4}$), $\frac{\Pi}{6}$ (or $\frac{5\Pi}{6}$). Note that $\sigma_\alpha \sigma_\beta =$ rotation through 2θ.

(3) Show by example that $\alpha - \beta$ may be a root ($\alpha, \beta \in \Delta$) even when $(\alpha, \beta) \leq 0$.

(4) Let $\alpha, \beta \in \Delta$ span a subspace E' of E.
(a) Prove that $E' \cap \Delta$ is a root system in E'.
(b) Prove that $\Delta \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is a root system in E'.
(c) Is it necessarily true that these two root systems coincide?

(5) Let Δ' be a nonempty subset of Δ such that $\Delta' = -\Delta'$, and such that $\alpha, \beta \in \Delta'$, $\alpha + \beta \in \Delta$ implies that $\alpha + \beta \in \Delta'$. Prove that Δ' is a root system in the subspace of E that it spans.

25 May 2010