LIE ALGEBRAS: HOMEWORK 9
DUE: 8 JUNE 2010

Let E be a Euclidean space, and let Δ be a root system in E with Weyl group W. Let Π be a base for Δ.

(1) If $\sigma \in W$ can be written as a product of n simple reflections, prove that n has the same parity as $l(\sigma)$, (the length of σ).

(2) Define a function $sn : W \to \{\pm 1\}$ by $sn(\sigma) = (-1)^{l(\sigma)}$. Prove that sn is a homomorphism.

(3) Prove that the intersection of "positive" open half-spaces associated with any basis $x_1, \ldots, x_n$ of E is non-empty. (Hint: Let y_i be the projection of x_i on the orthogonal complement of the subspace spanned by all basis vectors except x_i, and consider $z = \sum r_i y_i$ when all $r_i > 0$.)

(4) Prove that there is a unique element σ in W sending Δ^+ to Δ^-. Prove that any reduced expression for σ must involve all σ_α with $\alpha \in \Pi$. Discuss $l(\sigma)$.

(5) Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a base for Δ. Let

$$\lambda = \sum_{i=1}^{r} k_i \alpha_i$$

with $k_i \in \mathbb{Z}$ and all $k_i \geq 0$ or all $k_i \leq 0$. Prove that either λ is a multiple (possibly zero) of a root, or else there exists $\sigma \in W$ such that $\sigma(\lambda) = \sum_{i=1}^{r} c_i \alpha_i$ with some $c_i > 0$ and some $c_i < 0$. (Sketch of proof: If λ is not a multiple of any root, then the hyperplane P_λ orthogonal to λ is not included in $\cup_{\alpha \in \Delta} P_\alpha$. Take $\mu \in P_\lambda - \cup_{\alpha \in \Delta} P_\alpha$. Then find $\sigma \in W$ for which all $(\alpha_i, \sigma(\mu)) > 0$. It follows that

$$0 = (\lambda, \mu) = (\sigma(\lambda), \sigma(\mu)) = \sum_{i=1}^{r} c_i (\alpha_i, \sigma(\mu)).$$

1 June 2010