LIE ALGEBRAS: LECTURE 10
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1. Irreducible root systems

Let E be a Euclidean space, and let Δ be a root system in E. We call a root system *irreducible* if it can not be partitioned into the union of two proper subsets $\Delta = \Delta_1 \cup \Delta_2$, such that $(\alpha, \beta) = 0$ for all $\alpha \in \Delta_1$ and $\beta \in \Delta_2$.

Example 1.1. A_2 is irreducible, but $A_1 \times A_1$ is not irreducible.

Lemma 1.2. *A root system Δ is irreducible if and only if Π can not be partitioned into the union of two proper subsets $\Pi = \Pi_1 \cup \Pi_2$, such that $(\alpha, \beta) = 0$ for all $\alpha \in \Pi_1$ and $\beta \in \Pi_2$.*

Proof. "$\Leftarrow$" Suppose that Π is irreducible and suppose that $\Delta = \Delta_1 \cup \Delta_2$ with $(\Delta_1, \Delta_2) = 0$. Let $\Pi_1 = \Pi \cap \Delta_1$ and $\Pi_2 = \Pi \cap \Delta_2$. Since $\Pi = \Pi_1 \cup \Pi_2$ with $(\Pi_1, \Pi_2) = 0$, we have without loss of generality that $\Pi_1 = \Pi$ and $\Pi_2 = \varnothing$. Then $\Pi \subset \Delta_1$. Since Π spans E, this implies that $(E, \Delta_2) = 0$. Hence, $\Delta_2 = \varnothing$.

"$\Rightarrow$" Suppose that Δ is irreducible and suppose that $\Pi = \Pi_1 \cup \Pi_2$ with $(\Pi_1, \Pi_2) = 0$. Now for each root $\alpha \in \Delta$ there exists $w \in W$ such that $w(\alpha)$ is a simple root. Let Δ_i be the set of roots conjugate to a root in Π_i. Then $\Delta = \Delta_1 \cup \Delta_2$. Now W is generated by simple reflections σ_α with $\alpha \in \Pi$. If $\alpha_i \in \Pi_i$, then $\sigma_{\alpha_i}(\alpha_j) = \alpha_j$ when $j \neq i$. Hence, each root of Δ_i is obtained from Π_i by adding or subtracting elements of Π_i. Therefore, $(\Delta_1, \Delta_2) = 0$. Since Δ is irreducible we conclude that either $\Delta_1 = \varnothing$ or $\Delta_2 = \varnothing$. Hence, $\Pi_1 = \varnothing$ or $\Pi_2 = \varnothing$. $\square$

Lemma 1.3. *Δ decomposes uniquely as the union of irreducible root systems $\Delta_i \subset E_i$ such that $E = E_1 \oplus \cdots \oplus E_m$.*

Proof. Let $\Pi = \Pi_1 \cup \cdots \cup \Pi_m$ be the decomposition of the base into mutually orthogonal irreducible subsets. Let E_i be the span of Π_i in E. Then $E = E_1 \oplus \cdots \oplus E_m$. Let $\Delta_i = W(\Pi_i)$. By a previous argument, $\Delta_i \subset E_i$, $(\Delta_i, \Delta_j) = 0$ when $i \neq j$, and $\Delta = \Delta_1 \cup \cdots \cup \Delta_m$. The Weyl group W_i of Δ_i is the subgroup of W generated by reflections σ_{α_i} with $\alpha_i \in \Pi_i$, and Δ_i is invariant under W_i. Thus, Δ_i is an irreducible root system in E_i. $\square$

Lemma 1.4. *Let Δ be an irreducible root system. Then W acts irreducibly on E. The W-orbit of a root α spans E.*

Proof. The span of the W-orbit of a root is a W invariant subspace, so the second statement follows from the first. Suppose E' is a nonzero subspace of E invariant under W (i.e. $\sigma(E') \subset E'$ for all $\sigma \in W$). Then the orthogonal complement E'' of E' is also W invariant. Now for each $\alpha \in \Delta$ either $\alpha \in E'$ or $E' \subset P_\alpha$. Thus $\alpha \notin E'$ implies $\alpha \in E''$. Hence, each root lies in one subspace or the other. Then $\Delta_1 = \Delta \cap E'$, $\Delta_2 = \Delta \cap E''$ is a partition of Δ into orthogonal subsets. Then without loss of generality, $\Delta_2 = \varnothing$ and $\Delta_1 = \Delta$. Since Δ spans E, we conclude that $E = E'$. $\square$

Lemma 1.5. *Let Δ be an irreducible root system. Then at most two root lengths occur in Δ, and all roots of a given length are conjugate under W.*

Proof. First note that if $\alpha, \beta \in \Delta$, then by the previous lemma not all $\sigma(\alpha)$ ($\sigma \in W$) can be orthogonal to β. The the possible ratios of squared root lengths are 1, 2, 3, 1/2, 1/3. If we had three root lengths, then we would have a ratio of 2/3 which is not possible.

Now suppose α and β have the same length. After replacing one by a W-conjugate, we may assume that they are not orthogonal (and not equal). Then $\langle \alpha, \beta \rangle = \langle \beta, \alpha, \rangle = \pm 1$. By possibly replacing β by $-\beta = \sigma_\beta(\beta)$, we may assume that $\langle \alpha, \beta \rangle = \langle \beta, \alpha, \rangle = 1$. Then

$$(\sigma_\alpha \sigma_\beta \sigma_\alpha)(\beta) = \sigma_\alpha \sigma_\beta(\beta - \alpha) = \sigma_\alpha(\beta - \alpha - \beta) = \alpha.$$

$\square$

When we have roots of two distinct lengths, then we refer to them as *short roots* or *long roots.*

2. Semisimple Lie algebras

Lemma 2.1. *Let $\mathfrak{g}$ be a simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Then the root system corresponding to $\mathfrak{h}$ is irreducible.*

Proof. Suppose Δ decomposes as $\Delta_1 \cup \Delta_2$ where $(\Delta_1, \Delta_2) = 0$. If $\alpha \in \Delta_1$ and $\beta \in \Delta_2$, then $(\alpha+\beta, \beta) = (\beta, \beta) \neq 0$ and $(\alpha+\beta, \alpha) = (\alpha, \alpha) \neq 0$ implies that $\alpha + \beta \notin \Delta$. Thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ when $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. Let L be the subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_\alpha$ with $\alpha \in \Delta_1$. If $L = \mathfrak{g}$ then $[\mathfrak{g}, \mathfrak{g}_\beta] = 0$ for $\beta \in \Delta_2$, which is a contradiction since $Z(\mathfrak{g}) = 0$ because $\mathfrak{g}$ is simple. Hence, L is a a proper subalgebra of $\mathfrak{g}$. The previous calculation also shows that $[L, \mathfrak{g}] \subset L$. Thus, L is a non-trivial ideal, which is a contradiction. $\square$

Corollary 2.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Δ. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ is the decomposition of $\mathfrak{g}$ into simple ideals, then $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$ and the corresponding (irreducible) root system Δ_i can be regarded as a subsystem of Δ such that $\Delta = \Delta_1 \cup \cdots \cup \Delta_m$ is the decomposition of Δ into irreducible components.*

Proof. If $\alpha \in \Delta_i$ we can extend α to $\mathfrak{h}$, $\alpha : \mathfrak{h} \to \mathbb{F}$, by letting $\alpha(\mathfrak{h}_j) = 0$ for $j \neq i$. Then $\alpha \in \Delta$ with $\mathfrak{g}_\alpha \subset \mathfrak{g}_i$. Conversely, if $\alpha \in \Delta$ then $[\mathfrak{h}_i, \mathfrak{g}_\alpha] \neq 0$ for some i, since otherwise $\mathfrak{h}$ would contain $\mathfrak{g}_\alpha$. Then $\mathfrak{g}_\alpha \subset \mathfrak{g}_i$, so $\alpha \,|_{\mathfrak{h}_i}$ is a root of $\mathfrak{g}_i$. $\square$

3. Cartan matrix

Let E be a Euclidean space and Δ a root system in E. Let W be the Weyl group of Δ. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a base for Δ. The matrix defined by $(\langle \alpha_i, \alpha_j \rangle)$ is called the *Cartan matrix* of Δ. The diagonal entries are equal to 2, and all other entries are non-positive integers.

Example 3.1. The Cartan matrices for root systems of rank 2 are:

$$A_1 \times A_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \; A_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; \; B_2 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; \; G_2 \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The Cartan matrix of Δ is independent of the choice of Π, since W acts transitively on the set of bases. The Cartan matrix of Δ depends on the choice of order of the simple roots, so it is defined up to a permutation of the index set.

Two root systems $\Delta \subset E$ and $\Delta' \subset E'$ are "isomorphic" if there exists a vector space isomorphism $\phi : E \to E'$ which maps Δ onto Δ' such that $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all roots $\alpha, \beta \in \Delta$.

The following proposition shows that a root system is determined by its Cartan matrix, up to isomorphism.

Proposition 3.2. *Let $\Delta' \subset E'$ be another root system, with base $\Pi' = \{\alpha'_1, \ldots, \alpha'_r\}$. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for all $1 \leq i, j \leq r$, then the bijection $\alpha_i \to \alpha'_i$ extends uniquely to an isomorphism $\phi : E \to E'$ mapping Δ onto Δ' and satisfying $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Delta$.*

Proof. Since Π is a basis for E and Π' is a basis for E', there is a unique vector space isomorphism $\phi : E \to E'$ sending α_i to α'_i for $1 \leq i \leq r$. If $\alpha, \beta \in \Pi$, then

$$\begin{aligned} \sigma_{\phi(\alpha)}(\phi(\beta)) &= \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \phi(\alpha) \\ &= \phi(\beta) - \langle \beta, \alpha \rangle \phi(\alpha) \\ &= \phi(\beta - \langle \beta, \alpha \rangle \alpha) \\ &= \phi(\sigma_\alpha(\beta)). \end{aligned}$$

So for each $\alpha \in \Pi$ the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\sigma_\alpha"] & E' \arrow[d, "\sigma_{\phi(\alpha)}"] \\ E \arrow[r, "\phi"] & E' \end{tikzcd} .$$

Since the Weyl groups W and W' are generated by simple reflections in Π and Π', respectively, the map

$$\sigma \to \phi \circ \sigma \circ \phi^{-1}$$

is an isomorphism of W onto W' sending σ_α to $\sigma_{\phi(\alpha)}$. Now each root $\beta \in \Delta$ is conjugate to a simple root, say $\sigma(\beta) = \alpha \in \Pi$ with $\sigma \in W$. Then

$$\phi(\beta) = (\phi \circ \sigma \circ \phi^{-1})(\phi(\alpha)) \in \Delta'.$$

Hence, $\phi(\Delta) \subset \Delta'$. By the same argument, $\phi^{-1}(\Delta') \subset \Delta$. Therefore, $\phi : E \to E'$ is an isomorphism mapping Δ onto Δ'. Finally, $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Delta$ follows from the fact that $\sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\sigma_\alpha(\beta))$ for all $\alpha, \beta \in \Delta$. □

4. Dynkin diagram

A *Coxeter group* is a group defined by a presentation of the form

$$\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. Then $m_{ij} = m_{ji}$. The condition $m_{ij} = \infty$ means that there is no relation of the form $(s_i s_j)^m = 1$.

This information can be encoded in the form of a *Coxeter graph*, where the vertices are labeled by the generators and the edges are determined by the coefficients m_{ij}. Two vertices s_i and s_j have an edge connecting them if $m_{ij} \geq 3$, and are labeled by m_{ij} if $m_{ij} \geq 4$.

Example 4.1. The Coxeter group

$$\langle s_1, s_2, s_3 \mid (s_i s_i) = 1,\ (s_1 s_2)^3 = 1,\ (s_1 s_3)^2 = 1,\ (s_2 s_3)^4 = 1 \rangle$$

has Coxeter graph: $s_1 \text{ —— } s_2 \overset{4}{\text{——}} s_3$.

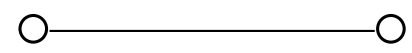
Let E be a Euclidean space and Δ a root system in E. Let W be the Weyl group of Δ. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a base for Δ. If α, β are distinct positive roots, then $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$.

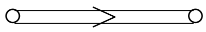
Define the *Coxeter diagram* of Δ to be the graph with r vertices corresponding to the simple roots, where vertex i is joined to vertex j by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. By the following method, we can convert a Coxeter diagram of Δ to a Coxeter graph whose Coxeter group is the Weyl group for Δ: replace every double edge with an edge labeled 4, and replace every triple edge with an edge labeled 6. Hence, the Coxeter diagram determines the Weyl group.

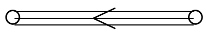
When $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 1$, then the roots α and β have the same length. However, if the vertices are joined by two or three edges, then the graph does not tell us which vertex corresponds to the long root and which corresponds to the short root. In this case, we draw an arrow pointing at the short root. The resulting diagram is called the *Dynkin diagram* of Δ. It encodes all the information of the Cartan matrix of Δ.

Example 4.2. The rank 2 Dynkin diagrams are:

$A_1 \times A_1$ 

A_2 

B_2 

G_2 

Since two vertices are not connected precisely when the corresponding simple roots are orthogonal to each other, the decomposition of the Dynkin diagram into connected components corresponds to the decomposition of Δ into irreducible root systems (and the decomposition of Π). Thus in order to classify irreducible root systems, it suffices to classify connected Dynkin diagrams.

Theorem 4.3. *If Δ is an irreducible root system of rank l, then its Dynkin diagram is one of the diagrams listed in Humphreys, Section 11.4. (page 58)*

Theorem 4.4. *For each Dynkin diagram of type A-G, there exists an irreducible root system having the given diagram. See Humphreys, Section 12.1 for the full description. (pages 64-65)*