LIE ALGEBRAS: LECTURE 12
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1. Root systems

Lemma 1.1. *Let Δ be a root system in a Euclidean space E and Π a base. If $\beta \in \Delta^+ \setminus \Pi$ then $\beta - \alpha \in \Delta^+$ for some $\alpha \in \Pi$*

Proof. If $(\beta, \alpha) \leq 0$ for all $\alpha \in \Pi$, then $\Pi \cup \{\beta\}$ would be a linearly independent set. So $(\beta, \alpha) > 0$ for some $\alpha \in \Pi$, and hence by a previous lemma $\beta - \alpha \in \Delta$. Since β is not proportional to α and α is simple, we conclude that $\beta - \alpha \in \Delta^+$. □

Corollary 1.2. *If $\beta \in \Delta^+$, then there exist $\alpha_i \in \Pi$, $i = 1, \dots s$, such that $\beta = \alpha_1 + \cdots + \alpha_s$ and each partial sum $\alpha_1 + \cdots + \alpha_k \in \Delta^+$, $(k \leq s)$.*

Proof. This is proven by induction on the height of β, using the previous lemma. □

2. Free Lie algebras

Let L be a Lie algebra generated by a set X. We say that L is *free on* X if, given any mapping $\phi : X \to M$ with M a Lie algebra, there exists a unique homomorphism $\psi : L \to M$ extending ϕ. Uniqueness is simple to verify. For existence, let V be a vector space with basis X. Let $\mathcal{T}(V)$ be the tensor algebra on V viewed as a Lie algebra via the bracket operation ($[x, y] := x \otimes y - y \otimes x$ for $x, y \in \mathcal{T}(V)$), and let L be the Lie subalgebra generated by X.

If L is a free Lie algebra on the set X, and K is an ideal of L generated by elements k_i $(i \in I)$, then we call the Lie algebra L/K the Lie algebra with *generators* x_j and *relations* $k_i = 0$, where x_j are the images of the elements of X in L/K.

3. Automorphisms

Let $\mathfrak{g}$ be a semisimple Lie algebra. An automorphism of $\mathfrak{g}$ is an isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$. An automorphism of the form

$$e^{\mathrm{ad}(x)} = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{ad}(x))^n,$$

$x \in \mathfrak{g}$ is called *inner*, and the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by these is denoted $\mathrm{Int}(\mathfrak{g})$ and its elements are called *inner automorphisms*. Note that $e^{\mathrm{ad}(x)}$ is well defined when $\mathrm{ad}(x)$ is nilpotent.

Let $\phi : \mathfrak{g} \to \mathfrak{g}$ be an isomorphism of $\mathfrak{g}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with root system Δ. If $\phi(\mathfrak{h}) = \mathfrak{h}$, then ϕ induces an automorphism of the root system Δ.

Every element w of the Weyl group W is induced by an inner automorphism of $\mathfrak{g}$ which leaves $\mathfrak{h}$ invariant. In particular, the reflection $\sigma_\alpha \in W$, $\alpha \in \Delta$, is induced by the element

$$e^{\mathrm{ad}(x_\alpha)} e^{-\mathrm{ad}(y_\alpha)} e^{\mathrm{ad}(x_\alpha)}.$$

4. Borel subalgebras

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$$

be the corresponding root space decomposition. Choose a set of simple roots Π and corresponding decomposition $\Delta = \Delta^+ \coprod \Delta^-$. Set

$$\mathfrak{n}^+ := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- := \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

Then one has a *triangular decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Also, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent subalgebras of $\mathfrak{g}$. Indeed, $\mathfrak{n}^+$ is a subalgebra, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and if $\alpha, \beta \in \Delta^+$ then either $\alpha + \beta \in \Delta_+$ or $\mathfrak{g}_{\alpha+\beta} = 0$.

To check nilpotence of $\mathfrak{n}^+$ recall that for $\alpha \in \Delta^+$ we have $\alpha = \sum_{\beta \in \Pi} r_\beta \beta$ with $r_\beta \in \mathbb{Z}_{\geq 0}$, and the height of α is $\mathrm{ht}(\alpha) = \sum_{\beta \in \Pi} r_\beta$. Also, for $\alpha_1, \alpha_2 \in \Delta$

we have $\mathrm{ht}(\alpha_1+\alpha_2) = \mathrm{ht}(\alpha_1)+\mathrm{ht}(\alpha_2)$. Set $\mathfrak{n}_1^+ := [\mathfrak{n}^+, \mathfrak{n}^+]$ and $\mathfrak{n}_{k+1}^+ := [\mathfrak{n}_k^+, \mathfrak{n}^+]$. Then

$$\mathfrak{n}_k^+ \subset \sum_{\alpha \in \Delta^+ : \mathrm{ht}(\alpha) > k} \mathfrak{g}_\alpha.$$

Since Δ^+ is finite, there exists an integer m such that $\mathrm{ht}(\alpha) < m$ for all $\alpha \in \Delta^+$. Thus $\mathfrak{n}_m^+ = 0$.

Set

$$\mathfrak{b}^+ := \mathfrak{n}^+ \oplus \mathfrak{h}, \qquad \mathfrak{b}^- := \mathfrak{n}^- \oplus \mathfrak{h}.$$

The subalgebra $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is solvable because $[\mathfrak{b}^+, \mathfrak{b}^+] = \mathfrak{n}^+$ (resp. $[\mathfrak{b}^-, \mathfrak{b}^-] = \mathfrak{n}^-$), which is nilpotent. The algebra $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is a *Borel subalgebra*: a maximal solvable subalgebra.

To see that $\mathfrak{b}^+$ is a maximal solvable subalgebra, first observe that if $S \supsetneq \mathfrak{b}^+ \supset \mathfrak{h}$ is a subalgebra then $\mathfrak{h}$ acts diagonally on S. Then $\mathfrak{g}_{-\alpha} \subset S$ for some $\alpha > 0$, implying that S contains a simple subalgebra isomorphic to $\mathfrak{sl}_2$. Thus S is not solvable.

5. Generators and relations

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Fix a base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ for the corresponding root system Δ. Recall that $\langle \alpha_j, \alpha_i \rangle = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} = \alpha_j(h_i)$, and the matrix A with entries $a_{ij} = \langle \alpha_j, \alpha_i \rangle$ is the Cartan matrix. For each $i = 1, \ldots, n$, choose $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$ such that $[x_i, y_i] = h_i$.

Theorem 5.1. (*a*) *$\mathfrak{n}^+$ is generated by the elements x_i, $\mathfrak{n}^-$ is generated by the elements y_i, and $\mathfrak{g}$ is generated by the elements x_i, y_i, h_i with $1 \le i \le n$.*

(*b*) *These elements satisfy the Weyl relations, $1 \le i, j \le n$:*

(1) $[h_i, h_j] = 0$;
(2) $[x_i, y_j] = \delta_{ij} h_i$;
(3) $[h_i, x_j] = a_{ij} x_j$, $[h_i, y_j] = -a_{ij} y_j$.

(*c*) *They also satisfy the Serre relations:*

(1) $ad(x_i)^{-a_{ij}+1} x_j = 0 \qquad (i \neq j)$;
(2) $ad(y_i)^{-a_{ij}+1} y_j = 0 \qquad (i \neq j)$.

Proof. (a) It suffices to show that $\mathfrak{n}^+$ is generated by the elements x_i. Let $\beta \in \Delta^+$. Write $\beta = \alpha_{i_1} + \cdots + \alpha_{i_s}$ such that partial sums $\alpha_{i_1} + \cdots + \alpha_{i_k}$ belong to Δ^+ for each $k \leq s$. Let $x_\beta = [x_{i_s}, [x_{i_{s-1}}, ...[x_{i_2}, x_{i-1}]]]$. This is nonzero since for any $\beta_1, \beta_2 \in \Delta$, $[\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}] = \mathfrak{g}_{\beta_1+\beta_2}$ (follows from the proof of Proposition 1.2 in Lecture 8). Since $\dim \mathfrak{g}_\beta = 1$ for $\beta \in \Delta$, $\mathfrak{g}_\beta = \mathbb{F}x_\beta$. Since $\mathfrak{n}^+$ is the sum spaces $\mathfrak{g}_\beta$ with $\beta \in \Delta^+$, we conclude that $\mathfrak{n}^+$ is generated by the elements x_i.

(b) Now $[x_i, y_j] = 0$ for $i \neq j$ since $\alpha_i - \alpha_j$ is not a root. Also, $[h_i, x_j] = \alpha_j(h_i)x_j = a_{ij}x_j$. The other relations are clear.

(c) Set $f_{ij} = \mathrm{ad}(x_i)^{-a_{ij}+1}x_j$. Then $f_{ij} \in \mathfrak{g}_{\alpha_j+\alpha_i-a_{ij}\alpha_i}$. But, $\alpha_j + (1 - a_{ij})\alpha_i = \sigma_{\alpha_i}(\alpha_j - \alpha_i)$. Since $\alpha_j - \alpha_i$ is not a root, neither is $\sigma_{\alpha_i}(\alpha_j - \alpha_i)$. Hence, $f_{ij} = 0$. The other relation is proved in the same manner. $\square$

Theorem 5.2. *The algebra $\mathfrak{g}$ is defined by the generators x_i, y_i, h_i, with $1 \leq i \leq n$, along with the Weyl relations and Serre relations.*

The proof of this theorem will be given at the end of the lesson.

6. Existence and Uniqueness

To read about the existence of a Cartan subalgebra and the conjugacy theorem for Cartan subalgebras, see Serre "Complex Semisimple Lie Algebras" Chapter 3. It follows that the root system of a semisimple Lie algebra is independent (up to isomorphism) of the chosen Cartan subalgebra.

Theorem 6.1. *Two semisimple Lie algebras with isomorphic root systems are isomorphic.*

Proof. Let $\mathfrak{g}$ (resp. $\mathfrak{g}'$) be a semisimple Lie algebra, $\mathfrak{h}$ (resp. $\mathfrak{h}'$) a Cartan subalgebra of $\mathfrak{g}$ (resp. $\mathfrak{g}'$), and Π (resp. Π') a base for the corresponding root system. Let $r : \Pi \to \Pi'$ be a bijection sending the Cartan matrix of Π to the Cartan matrix of Π'. For each $\alpha_i \in \Pi$ (resp. $\alpha'_j \in \Pi'$) let x_i (resp. x'_j) be a nonzero element of $\mathfrak{g}_{\alpha_i}$ (resp. $\mathfrak{g}'_{\alpha'_j}$). There is a a unique isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ sending h_i to $h'_{r(i)}$ and x_i to $x'_{r(i)}$ for all $\alpha_i \in \Pi$. Indeed, let y_i (resp. y'_j) be the element of $\mathfrak{g}_{-\alpha_i}$ (resp. $\mathfrak{g}_{-\alpha'_j}$) such that $[x_i, y_i] = h_i$ (resp. $[x'_j, y'_j] = h'_j$). Since $\Delta \cong \Delta'$, these generators satisfy the same relations.

Hence, Theorem 5.2 provides a unique homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$. This map is clearly surjective, since the generators of $\mathfrak{g}'$ are in the image. And since $\dim \mathfrak{g} = \dim \mathfrak{g}'$, we conclude that ϕ is an isomorphism. □

Theorem 6.2. *Let Δ be a root system. Then there exists a semisimple Lie algebra $\mathfrak{g}$ whose root system is isomorphic to Δ.*

Proof. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a base for Δ, with Cartan matrix A where $a_{ij} = \langle \alpha_j, \alpha_i \rangle$. Let $\mathfrak{g}$ be the Lie algebra defined by $3n$ generators x_i, y_i, h_i and by the Weyl and Serre relations. We will prove that this Lie algebra is finite dimensional, semisimple, and has root system isomorphic to Δ. □

Corollary 6.3. *For a semisimple Lie algebra $\mathfrak{g}$ to be simple, it is necessary and sufficient that Δ should be irreducible.*

Proof. We proved previously that a simple Lie algebra has an irreducible root system. Now suppose $\mathfrak{g}$ is a semisimple Lie algebra containing a proper ideal $\mathfrak{a}$. Then $\mathfrak{h}$ acts diagonally on $\mathfrak{a}$, so $\mathfrak{a} = \mathfrak{h}' \oplus (\oplus_{\alpha\in\Delta'}\mathfrak{g}_\alpha)$, where $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{a}$ and $\Delta' = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \cap \mathfrak{a} \neq \{0\}\}$. Let $\mathfrak{c}$ be the ideal of $\mathfrak{g}$ which is complement to $\mathfrak{a}$, $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{c}$. Then $\mathfrak{c} = \mathfrak{h}'' \oplus (\oplus_{\alpha\in\Delta''}\mathfrak{g}_\alpha)$, where $\mathfrak{h}'' = \mathfrak{h} \cap \mathfrak{c}$ and $\Delta'' = \Delta \setminus \Delta'$. Then $\Delta = \Delta' \cup \Delta''$ is a non-trivial decomposition of Δ into orthogonal sets. Indeed, let $\alpha \in \Delta'$ and $\beta \in \Delta''$. If $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Delta$. But $\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{a} \cap \mathfrak{c} = \{0\}$. If $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Delta$. But $\mathfrak{g}_{\alpha-\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\beta}] \subset \mathfrak{a} \cap \mathfrak{c} = \{0\}$ Hence, $(\alpha, \beta) = 0$. □

Now $\mathfrak{g}$ is simple $\Leftrightarrow$ Δ is irreducible $\Leftrightarrow$ Π is irreducible $\Leftrightarrow$ the Dynkin diagram is irreducible $\Leftrightarrow$ the Cartan matrix is irreducible. (A matrix A is *irreducible* if the index set I can not be non-trivially decomposed $I = I_1 \cup I_2$ such that $a_{ij} = a_{ji} = 0$ for all $i \in I_1$, $j \in I_2$.)

7. Serre's Theorem

Let Δ be a root system in a Euclidean space E, with positive definite symmetric bilinear form $(-,-)$. Let $\mathfrak{h} = E^*$, so that $E = \mathfrak{h}^*$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a base for Δ, and define $h_1, \dots, h_n \in \mathfrak{h}$ such that $\alpha_j(h_i) = \frac{2(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)} = \langle \alpha_j, \alpha_i \rangle$ for $1 \leq i, j \leq n$. Let A be the Cartan matrix of Δ, $a_{ij} = \langle \alpha_j, \alpha_i \rangle$.

Theorem 7.1. *Let $\mathfrak{g}$ be the Lie algebra defined by the $3n$ generators x_i, y_i, h_i, and by the relations $(1 \leq i, h \leq n)$:*

W1: $[h_i, h_j] = 0$;
W2: $[x_i, y_j] = \delta_{ij} h_i$;
W3: $[h_i, x_j] = a_{ij} x_j, \qquad [h_i, y_j] = -a_{ij} y_j$;
S1: $ad(x_i)^{-a_{ij}+1} x_j = 0 \qquad (i \neq j)$;
S2: $ad(y_i)^{-a_{ij}+1} y_j = 0 \qquad (i \neq j)$.

Then $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, with a Cartan subalgebra $\mathfrak{h}$ generated by the elements h_i, and its root system is Δ.

Proof. First consider the algebra $\mathfrak{a}$ defined by the $3n$ generators x_i, y_i, h_i, and by the Weyl relations W1-W3. Then $\mathfrak{a} = \mathfrak{m}^- \oplus \mathfrak{h} \oplus \mathfrak{m}^+$, where $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) is the free Lie algebra generated by x_i (resp. y_i), and $\mathfrak{h}$ has the elements h_i as a basis. (For a proof of this statement, see Humphreys Section 18.)

Now let $f_{ij}^+ = \mathrm{ad}(x_i)^{-a_{ij}+1} x_j$ and $f_{ij}^- = \mathrm{ad}(y_i)^{-a_{ij}+1} y_j$. Then $f_{ij}^+ \in \mathfrak{m}^+$ and $f_{ij}^- \in \mathfrak{m}^-$. Let $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$) denote the ideal of $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) generated by f_{ij}^+ (resp. f_{ij}^-).

(a): $\mathfrak{u}^+$, $\mathfrak{u}^-$, and $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ are ideals of $\mathfrak{a}$.

Let $\mathcal{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. The adjoint representation $\mathrm{ad} : \mathfrak{a} \to \mathrm{End}(\mathfrak{a})$ defines a $\mathcal{U}(\mathfrak{a})$-module structure on $\mathfrak{a}$. The ideal u_{ij} of $\mathfrak{a}$ generated by f_{ij}^+ is equal to the submodule $\mathcal{U}(\mathfrak{a}) \cdot f_{ij}^+$. By the PBW Theorem, u_{ij} is spanned by elements $XYH \cdot f_{ij}$ with $X \in \mathcal{U}(\mathfrak{m}^+)$, $Y \in \mathcal{U}(\mathfrak{m}^-)$, and $H \in \mathcal{U}(\mathfrak{h})$. Since $\mathrm{ad}(h_t)(f_{ij}^+) = \beta(h_t) f_{ij}^+$ with $\beta = \alpha_i + (1 - a_{ij})\alpha_j$, we have that $H \cdot f_{ij}^+$ is proportional to f_{ij}^+. A computation shows that $\mathrm{ad}(y_k)(f_{ij}^+) = 0$ for all k, thus $Y \cdot f_{ij}^+$ is proportional to f_{ij}^+. Hence, u_{ij} is generated by the elements $X \cdot f_{ij}^+$, and therefore is contained in $\mathfrak{u}^+$. Since $\mathfrak{u}^+ = \sum u_{ij}$ we conclude that $\mathfrak{u}^+$ is an ideal of $\mathfrak{a}$.

(b): $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^- = \mathfrak{m}^-/\mathfrak{u}^-$ and $\mathfrak{n}^+ = \mathfrak{m}^+/\mathfrak{u}^+$.

This is true since $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ is the ideal generated by f_{ij}^- and f_{ij}^+.

(c): The endomorphisms $\mathrm{ad}(x_i)$ and $\mathrm{ad}(y_i)$ of $\mathfrak{g}$ are *locally nilpotent*: for each element $z \in \mathfrak{g}$ there exists $k \in \mathbb{Z}^+$ such that $\mathrm{ad}(x_i)^k(z) = 0$, $\mathrm{ad}(y_i)^k(z) = 0$.

Let V_i be set of all $z \in \mathfrak{g}$ such that $\mathrm{ad}(x_i)^k(z) = 0$ for some integer k. Then a computation shows that V_i is a Lie subalgebra of $\mathfrak{g}$. Now V_i contains y_j $(1 \leq j \leq n)$ by the Weyl relations and contains x_j $(1 \leq j \leq n)$ by the Serre relations, and hence V_i contains $h_j = [x_j, y_j]$ $(1 \leq j \leq n)$. Since these generate $\mathfrak{g}$, we conclude that $V_i = \mathfrak{g}$.

Now if $\lambda \in \mathfrak{h}^*$, denote by $\mathfrak{a}_\lambda$ (resp. $\mathfrak{g}_\lambda$) the set of $z \in \mathfrak{a}$ (resp. $z \in \mathfrak{g}$) such that $\mathrm{ad}(h)z = \lambda(h)z$ for all $h \in \mathfrak{h}$. Then $\mathfrak{a}$ is direct sum of the weight spaces $\mathfrak{a}_\lambda$, since $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are free Lie algebras and $\mathfrak{h} = \mathfrak{a}_0$. The ideal $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ is generated by homogeneous elements, so the quotient Lie algebra $\mathfrak{g} = \mathfrak{a}/(\mathfrak{u}^+ \oplus \mathfrak{u}^-)$ is also a direct sum of weight spaces. Since $\mathfrak{a} = \mathfrak{m}^- \oplus \mathfrak{h} \oplus \mathfrak{m}^+$, we have that $\mathfrak{a}_\lambda \neq 0$ implies that λ is linear combination of simple roots with integer coefficients and all of the same sign. Thus the same is true for the quotient $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n}^+ = \oplus_{\lambda>0}\mathfrak{g}_\lambda$ and $\mathfrak{n}^- = \oplus_{\lambda<0}\mathfrak{g}_\lambda$. Next we want to find the dimension of each $\mathfrak{g}_\lambda$.

(d): If $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda = w(\mu)$ for some element w of the Weyl group W, then $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$.

It suffices to prove this when w is a simple reflection. For each $\alpha_i \in \Pi$, we define an automorphism of $\mathfrak{g}$ by $\phi_i = e^{\mathrm{ad}(x_i)}e^{-\mathrm{ad}(y_i)}e^{\mathrm{ad}(x_i)}$. Then ϕ_i induces the simple reflection σ_{α_i} on Δ, so ϕ_i sends $\mathfrak{g}_\mu$ to $\mathfrak{g}_\lambda$ if $\lambda = \sigma_{\alpha_i}(\mu)$ implying $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$.

(e): For $\alpha_i \in \Pi$, $\dim \mathfrak{g}_{\alpha_i} = 1$ and $\dim \mathfrak{g}_{m\alpha_i} = 0$ for $m \neq \pm 1, 0$.

This is clear for $\mathfrak{a}$, and since the ideal $\mathfrak{u}^+$ does not contain x_i, it is also true for $\mathfrak{g}$.

(f): If $\alpha \in \Delta$, then $\dim \mathfrak{g}_\alpha = 1$.

For each $\alpha \in \Delta$ there exists $w \in W$ such that $w(\alpha) \in \Pi$. This claim then follows from (d) and (e).

(g): Let λ be a linear combination of the simple roots α_i, with real coefficients, and suppose that λ is not a multiple of any root. Then there exists $w \in W$ such that $w(\lambda) = \sum t_i \alpha_i$ with some $t_i > 0$ and some $t_i < 0$. (This was a homework exercise.)

(h): If λ is not a root and $\lambda \neq 0$, then $\mathfrak{g}_\lambda = 0$.

λ is a linear combination of simple roots, since Π is a basis for E. If λ is a multiple of a root, then (h) follows from (d) and (e). Otherwise, there is some $w \in W$ such that $\mu = w(\lambda)$ is a linear combination of simple roots such that two coefficient have opposite signs. Then $\mathfrak{a}_\mu = 0$ implying $\mathfrak{g}_\mu = 0$. Applying (d) we have that $\mathfrak{g}_\lambda = 0$.

(i): The algebra $\mathfrak{g}$ has finite dimension, equal to $n + |\Delta|$.

By (f) and (h), we have that $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$ and for each $\alpha \in \Delta$ the dimension of $\mathfrak{g}_\alpha$ is one.

(j): If $\alpha \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha$ and the subalgebra $\mathfrak{s}_\alpha$ generated h_α, $\mathfrak{g}_\alpha$, and $\mathfrak{g}_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2$.

This is clear for simple roots, and follows by applying the automorphisms ϕ_i.

(k): $\mathfrak{g}$ is a semisimple Lie algebra.

Suppose that I is an abelian ideal in $\mathfrak{g}$. Then $\mathfrak{h}$ act diagonally on I, so $I = (I \cap \mathfrak{h}) \oplus \sum_{\alpha\in\Delta} I \cap \mathfrak{g}_\alpha$. If $I \cap \mathfrak{g}_\alpha \neq 0$ for some $\alpha \in \Delta$, then $\mathfrak{s}_\alpha \subset I$ since $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$. But we assumed that I is abelian, thus $I \subset \mathfrak{h}$. But $[I, x_j] = 0$ for all j implies that $\alpha_j(I) = 0$ for all j. Hence, $I = 0$.

(l): $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and Δ is the corresponding root system.

$\square$