# LIE ALGEBRAS: LECTURE 13
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## 1. Modules and weight spaces

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$$

be the corresponding root space decomposition. Choose a set of simple roots $\Pi$ and corresponding decomposition $\Delta = \Delta^+ \coprod \Delta^-$. Then one has a triangular decomposition, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+ = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{n}^- = \sum_{\alpha\in\Delta^-}\mathfrak{g}_\alpha$. Set

$$Q := \sum_{\alpha\in\Pi}\mathbb{Z}\alpha.$$

$Q$ is called the *root lattice*. Let $Q^+ = \sum_{\alpha\in\Pi}\mathbb{Z}_{\geq 0}\alpha$. Define a partial order on $\mathfrak{h}^*$ by $\lambda \geq \nu$ if $\lambda - \nu \in Q^+$.

Since $\mathfrak{h}$ acts diagonally on $\mathfrak{g}$ (via the adjoint action), $\mathfrak{h}$ acts diagonally on $\mathcal{U}(\mathfrak{g})$. Indeed, take an ordered eigenbasis $a_1, \dots, a_r$ in $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then the corresponding PBW basis of $\mathcal{U}(\mathfrak{g})$ is an eigenbasis with respect to $\mathfrak{h}$. If $a_i$ has weight $\mu_i$ for $1 \leq i \leq r$, then $a_1^{k_1}\cdots a_r^{k_r}$ has weight $\sum_{i=1}^r k_i\mu_i$.

Let $V$ be a $\mathfrak{g}$-module. We do not assume that $V$ is finite dimensional. Let $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$. If $V_\lambda \neq 0$, then $V_\lambda$ is called a *weight space* and $\lambda$ is called a *weight* of $V$. Set

$$V' = \sum_{\lambda\in\mathfrak{h}^*} V_\lambda.$$

**Lemma 1.1.**

(1) $\mathfrak{g}_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$, *(i.e.* $\mathfrak{g}_\alpha$ *maps* $V_\lambda$ *into* $V_{\lambda+\alpha}$*).*

(2) *The sum* $V' = \sum_{\lambda\in\mathfrak{h}} V_\lambda$ *is direct, and* $V'$ *is a* $\mathfrak{g}$*-submodule of* $V$*.*

(3) *If* $\dim V < \infty$*, then* $V = V'$*, (i.e.* $\mathfrak{h}$ *acts diagonally on* $V$*).*

*Proof.*

(1) Let $x \in \mathfrak{g}_\alpha$ and $v \in V_\lambda$. Then $h(xv) = x(hv) + [h,x]v = \lambda(h)xv - \alpha(h)xv = (\lambda+\alpha)(h)xv$. Thus, $xv \in V_{\lambda+\alpha}$. Therefore, $V'$ is stable under the action of $\mathfrak{g}$.

(2) The sum is direct because eigenvectors in different eigenspaces are linearly independent. By (1), $V'$ is stable under the action of $\mathfrak{g}$.

(3) By Weyl's Theorem, every finite dimensional $\mathfrak{g}$-module is a direct sum of simple modules. So it suffices to consider the case when $V$ is a simple $\mathfrak{g}$-module. Since $V$ is finite dimensional, it contains a non-zero vector $v$ which is an eigenvector for the action of $\mathfrak{h}$, by Lie's Theorem (since $\mathfrak{h}$ is solvable). Since $V$ is simple, $V = \mathcal{U}(\mathfrak{g})v$. Since $\mathfrak{h}$ acts diagonally on $\mathcal{U}(\mathfrak{g})$, the lemma follows.

$\square$

A $\mathfrak{g}$-module $V$ is called a *weight module* if $V = V'$. A $\mathfrak{g}$-module $V$ is called a *highest weight module* with *highest weight* $\mu$, if there is a $\mu \in \mathfrak{h}^*$ such that $V_\mu \neq 0$ and

$$V = \bigoplus_{\lambda \in \mathfrak{h}^* : \lambda \leq \mu} V_\lambda.$$

A non-zero vector $v^+ \in V_\lambda$ (for some $\lambda \in \mathfrak{h}^*$) of a $\mathfrak{g}$-module $V$ is called a *maximal vector* (of weight $\lambda$) if it is killed by $\mathfrak{g}_\alpha$ for all $\alpha > 0$, (i.e. $\mathcal{U}(\mathfrak{n}^+)v^+ = 0$). If $V$ is a highest weight module with highest weight $\mu$, then each non-zero vector $v \in V_\mu$ is a maximal vector. If a $\mathfrak{g}$-module $V$ is generated by a maximal vector $v^+ \in V_\lambda$ (i.e. $V = \mathcal{U}(\mathfrak{g})v^+$), then $V$ is called a *standard cyclic* module and $v^+$ is a *cyclic vector*.

## 2. Verma modules

Let $I(\lambda)$ be the left ideal in $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{n}^+$ and by the elements of the form $h - \lambda(h)1$ with $h \in \mathfrak{h}$. Then

$$M(\lambda) := \mathcal{U}(\mathfrak{g})/I(\lambda)$$

is the *Verma module* of highest weight $\lambda$. The quotient map $\phi : \mathcal{U}(\mathfrak{g}) \twoheadrightarrow M(\lambda)$ is a $\mathfrak{g}$-module homomorphism. Let $v_\lambda = \phi(1)$, where 1 is the identity element of $\mathcal{U}(\mathfrak{g})$. The Verma module $M(\lambda)$ is a standard cyclic module with cyclic vector $v_\lambda$. Indeed, $\mathfrak{n}^+ v_\lambda = 0$ and $v_\lambda$ has weight $\lambda$, since $(h - \lambda(h))v_\lambda = 0$ for all $h \in \mathfrak{h}$. The restriction of this map to $\mathcal{U}(\mathfrak{n}^-)$ is a $\mathcal{U}(\mathfrak{n}^-)$-module homomorphism. By the PBW theorem, this map $\mathcal{U}(\mathfrak{n}^-) \to M(\lambda)$ given by $u \mapsto uv_\lambda$ is bijective. Thus, $\mathcal{U}(\mathfrak{n}^-) \cong M(\lambda)$ as $\mathcal{U}(\mathfrak{n}^-)$-modules.

**Lemma 2.1.** *Any standard cyclic module is the quotient of a Verma module.*

*Proof.* Let $V$ be a standard cyclic module with cyclic vector $v^+ \in V_\lambda$. Then $\mathfrak{n}^+ v^+ = 0$ and $(h - \lambda(h))v^+ = 0$ for $h \in \mathfrak{h}$. So $I(\lambda)$ is in the kernel of the map $\mathcal{U}(\mathfrak{g}) \to V$ given by $u \mapsto uv^+$. Hence, this map factors through $M(\lambda)$. Since $v^+$ generates $V$, the induced map $M(\lambda) \to V$ is surjective. $\square$

**Lemma 2.2.** *Let $V$ be a standard cyclic $\mathfrak{g}$-module, with cyclic vector $v^+ \in V_\lambda$. Let $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$ and choose nonzero vectors $y_{\beta_i} \in \mathfrak{g}_{-\beta_i}$. Then:*

(1) *$V$ is spanned by the vectors $y_{\beta_1}^{k_1} \cdots y_{\beta_m}^{k_m} v^+$ $(k_i \in \mathbb{Z}_{\geq 0})$. In particular, $V$ is a highest weight module.*

(2) *The weights of $V$ are of the form $\nu = \lambda - \sum_{i=1}^n c_i\alpha_i$ $(c_i \in \mathbb{Z}_{\geq 0})$, where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Hence, all weights satisfy $\nu \leq \lambda$.*

(3) *For each $\nu \in \mathfrak{h}^*$, $\dim V_\nu < \infty$, and $\dim V_\lambda = 1$.*

(4) *Each submodule is a direct sum of its weight spaces.*

(5) *$V$ is an indecomposable $\mathfrak{g}$-module, with a unique maximal submodule and a corresponding unique irreducible quotient.*

*Proof.* Now $V$ is the quotient of the Verma module $M(\lambda)$, where $M(\lambda) \twoheadrightarrow V$ defined by $v_\lambda \mapsto v^+$ extended by the action of $\mathfrak{g}$. Thus (1), (2), and (3) follow from the fact that $\mathcal{U}(\mathfrak{n}^-) \cong M(\lambda)$ as $\mathcal{U}(\mathfrak{n}^-)$-modules, where $\mathcal{U}(\mathfrak{n}^-) \to M(\lambda)$ is defined by $u \mapsto uv_\lambda$. If $u \in \mathcal{U}(\mathfrak{n}^-)$ has weight $\sum_{\alpha_i \in \Pi} -c_i\alpha_i$, $c_i \in \mathbb{Z}_{\geq 0}$, then $uv^+$ has weight $\lambda - \sum_{\alpha_i \in \Pi} c_i\alpha_i$. Now (4) holds because eigenvectors in different eigenspaces are linearly independent.

For (5), let $S$ be the sum of the all the proper submodules of $M(\lambda)$. Then $S$ is clearly unique and maximal, but we must show that it is a proper submodule. Now each submodule is a direct sum of its weight spaces, and the sum of weight modules is again a weight module, with weight spaces $\left(\sum_{i\in I} V^{(i)}\right)_\mu = \sum_{i\in I} V_\mu^{(i)}$. To see that $S$ is a proper submodule, observe that $v^+ \notin P$ for any proper submodule $P$ implying $P_\lambda = 0$. Hence, $S_\lambda = 0$. It follows that $V$ is indecomposable, since if $V = M_1 \oplus M_2$ where $M_1$ and $M_2$ are proper submodules, then $M_1,\ M_2 \subset S$ implies $V = S$. □

**Corollary 2.3.** *Let $V$ be an irreducible standard cyclic $\mathfrak{g}$-module, with maximal vector $v^+ \in V_\lambda$. Then $v^+$ is the unique maximal vector in $V$, up to non-zero scalar multiples.*

*Proof.* Suppose $w^+ \in V_{\lambda'}$ is another maximal vector. Since $V$ is irreducible, $V = \mathcal{U}(\mathfrak{g})w^+$ implying that $w^+$ is also a cyclic vector. Then the second part of the lemma applies to both $\lambda$ and $\lambda'$. Hence $\lambda' \leq \lambda$ and $\lambda \leq \lambda'$ implies that $\lambda' = \lambda$. Since $\dim V_\lambda = 1$, we conclude that $w^+$ is proportional to $v^+$. □

**Corollary 2.4.** *Let $\overline{M}(\lambda)$ be the unique maximal submodule of the Verma module $M(\lambda)$. Then $V(\lambda) := M(\lambda)/\overline{M}(\lambda)$ is the unique simple quotient of $M(\lambda)$.*

**Corollary 2.5.** *If $V$ is a finite dimensional simple module, the $V \cong V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

*Proof.* The set of weights of $V$ is finite and so it contains a maximal element $\lambda$ under the partial ordering. Let $v'$ be a non-zero element of $V_\lambda$. Since $\lambda$ is a maximal weight, Lemma 1.1 implies that $\mathfrak{n}^+v' = 0$. Hence, $v'$ is a maximal vector. Thus the map $\phi : M(\lambda) \to V$ given by $v_\lambda \mapsto v'$ is a $\mathcal{U}(\mathfrak{g})$-module homomorphism. Since $V$ is simple, this map is surjective. Thus $V \cong M(\lambda)/\ker\phi$. Since $V(\lambda)$ is the unique simple quotient of $M(\lambda)$, we conclude that $V \cong V(\lambda)$. □

Let $\alpha \in \Delta$, and take $S_\alpha \cong \mathfrak{sl}(2)$ with generators $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{h}^*$. Recall that $< \lambda, \alpha >= \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = \lambda(h_\alpha)$.

**Theorem 2.6.** *If $V$ is a finite dimensional irreducible $\mathfrak{g}$-module of highest weight $\lambda$, then $< \lambda, \alpha > \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta^+$.*

*Proof.* Let $\alpha \in \Delta^+$. Then $V$ is a finite dimensional $S_\alpha$-module, with a maximal vector $v_\lambda$ of weight $\lambda$. Since $S_\alpha \cong \mathfrak{sl}_2$ is simple, we have by Weyl's Theorem that $V$ decomposes into a direct sum of irreducible $S_\alpha$-modules. The maximal weight of the irreducible $S_\alpha$-submodule $M$ that $v_\lambda$ generates must be a non-negative integer. Now $x_\alpha v_\lambda = 0$ and $h_\alpha v_\lambda = \lambda(h_\alpha) v_\lambda$. So the maximal weight of the $S_\alpha$-submodule $M$ is $\lambda(h_\alpha)$. Thus, $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$. Since $\alpha \in \Delta^+$ was chosen arbitrarily, we conclude that $< \lambda, \alpha > \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta^+$. $\square$

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots. Define the *weight lattice* to be

$$\begin{aligned} P(\Pi) :=& \{\lambda \in \mathfrak{h}^* \mid < \lambda, \alpha > \in \mathbb{Z} \text{ for all } \alpha \in \Delta^+\} \\ =& \{\lambda \in \mathfrak{h}^* \mid < \lambda, \alpha > \in \mathbb{Z} \text{ for all } \alpha \in \Pi\}. \end{aligned}$$

Elements of $P(\Pi)$ are called *integral weights.* The *dominant integral weights* are the following:

$$P^+(\Pi) = \{\lambda \in \mathfrak{h}^* \mid < \lambda, \alpha > \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Pi\}.$$