LIE ALGEBRAS: LECTURE 2
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CRYSTAL HOYT

1. Representations

Let L be a Lie algebra. In this course, we shall restrict our attention to the case that $\dim L$ is finite. A *representation* of a Lie algebra L is a homomorphism

$$\phi : L \to \mathfrak{gl}(V).$$

A representation is called *faithful* if the kernel of ϕ is trivial. The dimension of a representation is by definition the dimension of the vector space V.

Example 1.1. $\dim L = 1$. Then $L = ke$, and $\phi : L \to \mathfrak{gl}(V)$ is determined by the image of e. A representation is given by an endomorphism of V, $\phi(e)$.

Example 1.2. *L is commutative.* If $L = \langle e_1, \dots, e_n \rangle$, then a homomorphism $\phi : L \to \mathfrak{gl}(V)$ is determined by the images $\phi(e_i)$. It is necessary and sufficient that the endomorphisms $\phi(e_i)$ commute in order for ϕ to define a homomorphism.

Example 1.3. *Natural representation.* The Lie algebra $\mathfrak{gl}_n$ admits an n-dimensional representation, which is given by the identity map

$$id : \mathfrak{gl}_n \to \mathfrak{gl}(k^n).$$

Similarly, if $L \subseteq \mathfrak{gl}_n$ we have a natural n-dimensional representation of $\mathfrak{g}$.

Let L be a Lie algebra, and $x \in L$. Define a linear transformation

$$\mathrm{ad}_x : L \to L$$

by the formula $\mathrm{ad}_x(y) = [x, y]$.

Lemma 1.4. $\mathrm{ad}_x \in \mathrm{Der}\, L$, *i.e.* ad_x *is a derivation.*

Proof. The fact that ad_x is a linear endomorphism follows from the linearity of $[-,-]$. The fact that the Leibniz rule holds follows from the Jacobi identity and anticommutativity of the bracket. Indeed, for all $x, y, z \in L$,

$$\begin{aligned} \mathrm{ad}_x[y,z] &= [x,[y,z]] \\ &= [[x,y],z] + [y,[x,z]] \\ &= [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]. \end{aligned}$$

□

Derivations of the form ad_x are called *inner derivations*, and all others are called *outer derivations.*

Lemma 1.5. *Let L be a Lie algebra, and define a map*

$$\begin{aligned} ad :&L \to \mathrm{Der}(L) \\ &x \mapsto \mathrm{ad}_x\,. \end{aligned}$$

The map $\mathrm{ad} : L \to \mathrm{Der}(L)$ *is a homomorphism of Lie algebras.*

Proof. One simply has to check that

$$\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x,$$

which follows from the Jacobi identity. □

The map $\mathrm{ad} : L \to \mathrm{Der}(L)$ is called the *adjoint representation* of L, and is indeed a representation since $\mathrm{Der}(L) \subseteq \mathfrak{gl}(L)$ induces a map $\mathrm{ad} : L \to \mathfrak{gl}(L)$. The *center* of a Lie algebra L is defined as

$$Z(L) = \{z \in L \mid [z,x] = 0 \text{ for all } x \in L\}.$$

By the definition of the map ad, one has that $Z(L) = \mathrm{Ker}(\mathrm{ad})$. Hence, the adjoint representation of a Lie algebra L is faithful if and only if $Z(L) = 0$.

L is abelian if and only if $Z(L) = L$. The center of a Lie algebra L is an ideal in L. Another important ideal of L is the *derived algebra* of L, which is defined as

$$[L,L] = \mathrm{span}\{[x,y] \mid x, y \in L\}.$$

L is abelian if and only if $[L,L] = 0$. The quotient $L/[L,L]$ is an abelian Lie algebra.

Lemma 1.6. *If $X \in [L, L]$ and $\phi : L \to \mathfrak{gl}_n$ is a representation, then* $\operatorname{Tr} \phi(X) = 0$.

Proof. Indeed, write $X = \sum [A_i, B_i]$, then

$$\operatorname{Tr}(\phi(X)) = \operatorname{Tr}(\phi(\sum [A_i, B_i])) = \sum \operatorname{Tr}([\phi(A_i), \phi(B_i)]) = 0,$$

since $\operatorname{Tr}(ab) = \operatorname{Tr}(ba)$ for all $a, b \in \mathfrak{gl}_n$. □

Example 1.7. If $L = \mathfrak{gl}_n$, then $[L, L] = \mathfrak{sl}_n$. So, if $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_n$, then $[\mathfrak{g}, \mathfrak{g}]$ is a subalgebra of $\mathfrak{sl}_n$.

Let L be a Lie algebra. The *normalizer* of a subalgebra K of L is defined by

$$N_L(K) = \{x \in L \mid [x, K] \subseteq K\}.$$

By the Jacobi identity, $N_L(K)$ is a subalgebra of L. In particular, it is the largest subalgebra of L which contains K as an ideal. If $K = N_L(K)$, we call K *self normalizing*. The *centralizer* of a subset X of L is

$$C_L(X) = \{y \in L \mid [y, X] = 0.\}$$

By the Jacobi identity, $C_L(X)$ is a subalgebra of L. For example, $C_L(L) = Z(L)$.

2. Nilpotent Lie algebras

Define a sequence of ideals of L (the *lower central series*) by

$$D_1 L := [L, L], \qquad D_2 L := [L, D_1 L], \qquad D_k L := [L, D_{k-1} L].$$

L is called *nilpotent* if $D_n L = 0$ for some n.

Example 2.1. The Lie algebra $\mathfrak{n}_n$ of strictly upper triangular matrices is nilpotent. $\mathfrak{n}_n = \{a \in \mathfrak{gl}_n \mid a_{ij} = 0 \text{ for } i \geq j\}$

Proposition 2.2. *Let L be a Lie algebra.*

1. *If L is nilpotent, then so are all subalgebras and homomorphic images of L.*
2. *If $L/Z(L)$ is nilpotent, then so is L.*
3. *If L is nilpotent and nonzero, then $Z(L) \neq 0$.*

Proof. (1)If K is a subalgebra of L, then $D_nK \subseteq D_nL$. If $\phi : L \to M$ is an epimorphism, then $\phi(D_nL) = D_nM$. This follows from $\phi([L, L]) = [\phi(L), \phi(L)]$ using induction.
(2)If $L/Z(L)$ is nilpotent, then for some n $D_nL \subseteq Z(L)$. But then $D_{n+1} = [L, D_n] \subseteq [L, Z(L)] = 0$.
(3) If L is nilpotent with $D_nL = 0$ and $D_{n-1}L \neq 0$, then $D_{n-1}L \subseteq Z(L)$. □

An element $x \in L$ is called *ad-nilpotent* if $\operatorname{ad} x$ is a nilpotent endomorphism, i.e. $(\operatorname{ad} x)^n = 0$ for some n.

Lemma 2.3. *If $x \in \mathfrak{gl}(V)$ is a nilpotent linear endomorphism of V, then x is ad-nilpotent.*

Proof. Let $x \in \mathfrak{gl}(V)$ be a nilpotent linear endomorphism. Then $x^n = 0$ for some n. Define two linear endomorphisms of $\mathfrak{gl}(V)$ as follows:

$$L : \mathfrak{gl}(V) \to \mathfrak{gl}(V) \qquad\qquad R : \mathfrak{gl}(V) \to \mathfrak{gl}(V).$$
$$y \longmapsto xy \qquad\qquad y \longmapsto yx$$

Since $LR(y) = xyx = RL(y)$, we have that L and R commute. Since $L^n(y) = x^n y$ and $R^n(y) = yx^n$, we have that $L^n = 0$ and $R^n = 0$. Hence, L and R are nilpotent. Since $\operatorname{ad} x = L - R$, we conclude

$$(\operatorname{ad} x)^{2n} = (L - R)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} L^{2n-k}(-R)^k = 0.$$

Therefore, x is ad-nilpotent. □

Note that $id \in \mathfrak{gl}(V)$ is ad-nilpotent, and is clearly not nilpotent.

Lemma 2.4. $D_nL = 0$ *if and only if for any $x_0, \ldots, x_n \in L$,*

$$[x_n, [..[x_2[x_1, x_0]]]] = (\operatorname{ad} x_n)(\operatorname{ad} x_{n-1}) \ldots (\operatorname{ad} x_1)(x_0) = 0.$$

This lemma implies that any element of a nilpotent Lie algebra is ad-nilpotent. The converse of this statement is the following theorem.

Theorem 2.5. ***Engel's Theorem.*** *If a finite dimensional Lie algebra consists of ad-nilpotent elements, then it is nilpotent.*

First we will prove:

Theorem 2.6. *Let L be a subalgebra of $\mathfrak{gl}(V)$, where V is a non-zero finite dimensional vector space (over k). If L consists of nilpotent endomorphisms, then there exists a non-zero $v \in V$ for which $Lv = 0$, ($Lv = \{xv \mid x \in L\}$).*

Proof. We prove this by induction on $\dim L$. If $\dim L = 1$, then $L = kx$ where x is nilpotent. Then any non-zero $v \in \operatorname{Ker} x$ satisfies $Lv = 0$. Let K be a proper subalgebra of L. Then K acts on the vector space L via the adjoint action (i.e. $ad : K \to \mathfrak{gl}(L)$), and by Lemma 2.3, the elements of K act nilpotently. This induces an action of K on the vector space L/K, given as follows. For each $y \in K$, we have a map

$$\begin{aligned} \phi(y) : L/K &\longrightarrow L/K \\ x + K &\mapsto [y, x] + K. \end{aligned}$$

This map is well-defined since K is a subalgebra of L, and the map is linear by the linearity of $[,]$. The map $\phi : K \to \mathfrak{gl}(L/K)$ is an homomorphism. This can be checked using the Jacobi identity. The elements of K act nilpotently the vector space L/K, since they act nilpotently on L by the adjoint action.

By applying the induction hypothesis to $\phi(K) \subset \mathfrak{gl}(L/K)$, there exists a vector $x+K \neq K$ in L/K which is killed by the action of K. In other words, there exists $x \in L$ such that $x \notin K$ and $[x, K] \subseteq K$. Hence, $K' = K + kx$ is a subalgebra of L containing K as an ideal. Thus, any subalgebra of L is an ideal with codimension 1 in another subalgebra of L. Since L is finite dimensional, this implies that L has an ideal I of codimension 1: $L = I \oplus kx$.

By induction hypothesis, the space $V' := \{v \in V \mid Iv = 0\}$ is non-zero. The space V' is x-invariant, since for any $y \in I$, $v \in V'$ we have:

$$yxv = xyv + [y, x]v = 0.$$

Since x is nilpotent, we have a non-zero vector $v \in V'$ which is killed by x. Hence $Lv = 0$. □

Theorem 2.7. ***Engel's Theorem.*** *Let L be a finite dimensional Lie algebra. If all elements of L are ad-nilpotent, then L is nilpotent.*

Proof. Let L be a finite dimensional Lie algebra with all elements ad-nilpotent. Hence, the Lie algebra $\operatorname{ad} L \subset \mathfrak{gl}(L)$ satisfies the hypothesis of Theorem 2.6.

Thus, there exists an $x \in L$ such that $[L, x] = 0$, and so $Z(L) \neq 0$. Denote $Z = Z(L)$. Let $x + Z,\ y + Z \in L/Z$. Then

$$\begin{aligned}(\operatorname{ad}(x+Z))^n(y+Z) &= [x+Z, [x+Z, \ldots [x+Z, y+Z]]] \\ &= [x, [x, \ldots [x, y]]] + Z \\ &= (\operatorname{ad} x)^n(y) + Z.\end{aligned}$$

Thus, $L/Z(L)$ consists of ad-nilpotent elements and has smaller dimension than L. So by induction $L/Z(L)$ is nilpotent. By part (b) of Proposition 2.2, we conclude that L is nilpotent. □

Let V be a finite dimensional vector space, say $\dim V = n$. A *flag* in V is a chain of subspaces

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V,$$

where $\dim V_i = i$. If $x \in \operatorname{End} V$, we say x *stabilizes* this flag if $xV_i \subset V_i$ for all i.

Corollary 2.8. *Let L be a subalgebra of $\mathfrak{gl}(V)$, where V is a non-zero finite dimensional vector space (over k). If L consists of nilpotent endomorphisms, then there exists a flag (V_i) in V stable under L, with $xV_i \subset V_{i-1}$ for all i. In other words, there exists a basis of V relative to which L is a subalgebra of $\mathfrak{n}_n$, where $n = \dim V$.*

Proof. We prove this by induction on the dimension of V. Let $v_1 \in V$ be any non-zero vector such that $Lv_1 = 0$ (existence is given by Theorem 2.6). Set $V_1 = kv_1$. Let $W = V/V_1$. Let $x \in L \subset \mathfrak{gl}(V)$. Since $V_1 \in \operatorname{Ker} x$, we have a well-defined linear map $\bar{x} : V/V_1 \to V$. Composing this with the projection map $\rho : V \to V/V_1$, we obtain a map $\phi(x) \in \mathfrak{gl}(W)$. Since x is a nilpotent endomorphism, $\phi(x)$ is nilpotent. Thus the subalgebra $\phi(L) \subset \mathfrak{gl}(W)$ consists of nilpotent endomorphisms, and $\dim W < \dim V$. By induction hypothesis, there exists a flag in W stable under L and satisfying $xW_i \subset W_{i-1}$ for all i. Then

$$\{0\} \subset V_1 \subset \rho^{-1}(W_1) \subset \ldots \subset \rho^{-1}(W_{n-1})$$

is a flag in V stable under L which satisfies $xV_i \subset V_{i-1}$ for all i. □