LIE ALGEBRAS: LECTURE 3
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1. Simple 3-dimensional Lie algebras

Suppose L is a simple 3-dimensional Lie algebra over k, where k is algebraically closed. Then $[L, L] = L$, since otherwise either $[L, L]$ would be a non-trivial ideal or L would be abelian. Also, L is not nilpotent, because otherwise $Z(L)$ would be a non-trivial ideal in L. So, by Engel's Theorem, not every element of L is ad-nilpotent.

Choose $H \in L$ such that $\operatorname{ad} H$ is not nilpotent. Then $\operatorname{ad} H$ has a non-zero eigenvalue $\lambda \in k$ and a non-zero eigenvector $X \in L$, so that $[H, X] = \lambda X$. Since $[H, H] = 0$, H is an eigenvector of $\operatorname{ad} H$ with eigenvalue zero. Now $\operatorname{Tr}(\operatorname{ad} H) = 0$, because $H \in [L, L] = L$. We choose a basis for L such that $\operatorname{ad}H$ is in Jordan normal form. Using the fact that the dimension of L is 3, we see that the sum of the eigenvalues of $\operatorname{ad} H$ is zero. So there exists an eigenvector Y for $\operatorname{ad} H$ with eigenvalue $-\lambda$.

Since X, Y, H is an eigenbasis for the operator $\operatorname{ad} H$, it is a basis for L. It remains for us to express $[X, Y]$ in terms of this basis, say

$$[X, Y] = aX + bY + cH, \text{ with } a, b, c \in k.$$

Now by the Jacobi identity,

$$[H, [X, Y]] = [[H, X], Y] + [X, [H, Y]] = \lambda[X, Y] - \lambda[X, Y] = 0.$$

Also, $[H, [X, Y]] = [H, aX + bY + cH] = a\lambda X - b\lambda Y$, which implies that $a, b = 0$ and $[X, Y] = cH$. Since $[L, L] = L$, c must be non-zero. We may rescale H so that $\lambda = 2$, and we may then rescale X or Y so that $c = 1$. So we see that if k is algebraically closed, then the only simple 3 dimensional Lie algebra up to isomorphism is $\mathfrak{sl}_2(k)$.

If $k = \mathbb{R}$, then it is possible that the eigenvalue λ of $\operatorname{ad}H$ is not in k. If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then the complex conjugates λ and $\overline{\lambda}$ would both be eigenvalues of the real matrix $\operatorname{ad} H$. Since $\operatorname{Tr}(\operatorname{ad} H) = 0$, we must have $\lambda = i$ and $\overline{\lambda} = -i$ (after rescaling H). So over $\mathbb{C}$,

$$\operatorname{ad} H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then over $\mathbb{R}$ in some basis,

$$\operatorname{ad} H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So in some basis H, X, Y we have that $[H, X] = Y$ and $[H, Y] = -X$. And as before, we obtain $[X, Y] = cH$ for some non-zero $c \in \mathbb{R}$. Then by multiplying X and Y by an appropriate real scalar, we may assume that c is ± 1.

If $c = -1$, then we are in the previous case, since

$$\begin{aligned} &[Y, X + H] = X + H, \\ &[Y, X - H] = -(X - H), \\ &[X + H, X - H] = 2Y. \end{aligned}$$

If $c = 1$, then we have

$$[H, X] = Y, \qquad [X, Y] = H, \qquad [Y, H] = X,$$

which we recognize as the Lie algebra on $\mathbb{R}^3$ with $[,]$ given by the vector product. This Lie algebra L is not isomorphic to $\mathfrak{sl}_2$ when $k = \mathbb{R}$, because there does not exist an $A \in L$ such that $\operatorname{ad} A$ has a real non-zero eigenvalue. Hence, when $k = \mathbb{R}$ we have two distinct simple 3-dimensional Lie algebras.

2. Lemma

Lemma 2.1. *Let A and B be ideals in a Lie algebra L. Then*

$$[A, B] := \operatorname{span}\{[a, b] \mid a \in A,\ b \in B\}$$

is an ideal in L.

Proof. The set $[A, B]$ is a vector subspace by definition, so we only need to check that $[L, [A, B]] \subseteq [A, B]$. Let $[x, [a, b]] \in [L, [A, B]]$, with $x \in L$, $a \in A$ and $b \in B$. By the Jacobi identity,

$$[x, [a, b]] = [[x, a], b] + [a, [x, b]].$$

Since A and B are ideals, we have that $[x, a] \in A$ and $[x, b] \in B$. Hence, $[x, [a, b]] \in [A, B]$. Since $[L, [A, B]]$ is spanned by elements of this form, we conclude that $[L, [A, B]] \subseteq [A, B]$. $\square$

3. Solvable Lie algebras

Define a sequence of ideals of L (the derived series) by

$$D^1L := [L, L], \qquad D^2L := [D^1L, D^1L], \qquad D^kL := [D^{k-1}L, D^{k-1}L].$$

L is called solvable if $D^nL = 0$ for some n.

Example 3.1. Nilpotent Lie algebras are solvable, because $D^iL \subseteq D_iL$ for all i.

Example 3.2. The Lie algebra $\mathfrak{b}_n$ of upper triangular matrices is solvable, but is not nilpotent. $\mathfrak{b}_n = \{a \in \mathfrak{gl}_n \mid a_{ij} = 0 \text{ for } i > j\}$

Observe that if $[L, L]$ is solvable then L is solvable, since they share the same sequence of ideals. This statement is not true if we replace the word "solvable" with "nilpotent". A counter example is $[\mathfrak{b}_n, \mathfrak{b}_n] = \mathfrak{n}_n$ which is nilpotent, but $\mathfrak{b}_n$ is not nilpotent.

Proposition 3.3. *Let L be a Lie algebra.*

(1) *If L is solvable, then so are all subalgebras and homomorphic images.*
(2) *If I is a solvable ideal of L such that L/I is solvable, then L is solvable.*
(3) *If I, J are solvable ideals of L, then so is $I + J$.*

Proof. (1)If K is a subalgebra of L, then $D^nK \subseteq D^nL$. If $\phi : L \to M$ is an epimorphism, then $\phi(D^nL) = D^nM$. This follows from $\phi([L, L]) = [\phi(L), \phi(L)]$ using induction.
(2) Suppose $D^n(L/I) = 0$. Let $\pi : L \to L/I$ be the canonical projection map. Then $\pi(D^nL) = 0$ implies $D^nL \subseteq I = \operatorname{Ker} \pi$. Then if $D^mI = 0$ we conclude that $D^{m+n}L = D^m(D^nL) = 0$.
(3) If I, J are ideals, then one can check that $I + J := \{a + b \in L | a \in I,\ b \in J\}$ is an ideal. Note that $I, J \subset I + J$. By a standard isomorphism theorem

$$(I + J)/J \cong I/(I \cap J).$$

Suppose that I, J are solvable ideals of L. Then right hand side is solvable since it is a homomorphic image of I. Hence $(I + J)/J$ is solvable, and by part (2) we conclude that $I + J$ is solvable. $\square$

Corollary 3.4. *A Lie algebra L is solvable (nilpotent) if and only if* $\operatorname{ad} L$ *is solvable (nilpotent).*

Proof. "$\Rightarrow$" follows directly from part (1), since $\operatorname{ad} L$ is the image of the map ad. For "$\Leftarrow$", observe that $\operatorname{ad} L \cong L/\operatorname{Ker}(\operatorname{ad}) = L/Z(L)$. The claim then follows from part (2) and Proposition 2.2 (2), since the center $Z(L)$ is abelian, and hence solvable. $\square$

The third part of this lemma yields the existence of a unique maximal solvable ideal, called the *radical* of L and denoted $\operatorname{Rad} L$. If $\operatorname{Rad} L = 0$, then L is called *semisimple.*

The quotient algebra $L/\operatorname{Rad} L$ is semisimple. A semisimple Lie algebra has a trivial center. Hence, the adjoint representation of a semisimple Lie algebra is faithful.

4. Lie's Theorem

In this course, we will now assume that our field $\mathbb{F}$ is algebraically closed and has characteristic zero, unless explicitly stated otherwise. We saw an example illustrating the fact that working over $\mathbb{R}$ is not the same as working over $\mathbb{C}$.

Theorem 4.1 (Lie's Theorem)**.** *Assume that the field $\mathbb{F}$ is algebraically closed and has characteristic zero. Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, with* $\dim V = n$ *where n is non-zero and finite. Then V contains a common eigenvector for all endomorphisms in L.*

Thus, there exists a flag (V_i) in V which is stabilized by L, i.e. for all $x \in L$ we have hat $xV_i \subseteq V_i$ for all i. In other words, there exists a basis of V relative to which L is a subalgebra of $\mathfrak{b}_n$, where $n = \dim V$.

Proof. The second statement follows directly from the first. We prove the first statement by using induction on the dimension of L. If $\dim L = 1$, then $L = \mathbb{F}x$. Since $\mathbb{F}$ is algebraically closed and $\dim V$ is finite, x has an eigenvector v in V. Then v is an eigenvector for all elements of L. Now suppose that assertion holds when $\dim L < m$.

First, we find an ideal of codimension 1. Since L is solvable, $[L, L] \neq L$. Now any subspace of L which contains $[L, L]$ is an ideal in L. Hence, L has an ideal of codimension 1: $L = I \oplus \mathbb{F}Z$.

Now by induction hypothesis, there exists a common eigenvector $v \in V$ for all elements of I. So there is a linear function $\lambda : I \to \mathbb{F}$ such that for every $y \in I$ we have $y.v = \lambda(y)v$. Let

$$W = \{w \in V \mid y.w = \lambda(y)w \text{ for all } y \in I\}.$$

Then $v \in W$, so $W \neq 0$. One can show that W is a vector subspace of L.

We will show that L stabilizes W. Fix $x \in L$ and $w \in W$. We need to prove that $x.w \in W$. In particular, we need to show that for all $y \in I$, $y.(x.w) = \lambda(y)x.w$. Now

$$y.(x.w) = x.(y.w) - [x, y].w = \lambda(y)x.w - \lambda([x, y])w,$$

since $[x, y] \in I$. Thus, we need to prove that $\lambda([x, y]) = 0$, for all $y \in I$. For each $k \in \mathbb{N}$, let W_k denote the span of $w, x.w, \ldots, x^{k-1}.w$. Set $W_0 = 0$. Let $n > 0$ be the smallest integer for which $W_n = W_{n+1}$. Then x maps W_n into W_n. Moreover $\dim W_k = k$ for $k \leq n$.

We claim that for all $y \in I$,

$$y.x^k.w = \lambda(y)x^k.w \pmod{W_k},$$

and we prove this by induction on k. For $k = 0$, this follows from the definition of W. Suppose the claim holds for $k - 1$. Now

$$yx^k w = xyx^{k-1}w - [x, y]x^{k-1}w.$$

Since I is an ideal, we have $[x, y] \in I$. Then by induction hypothesis,

$$[x, y]x^{k-1}w = \lambda([x, y])x^{k-1}w (\operatorname{mod} W_{k-1})$$

and

$$yx^{k-1}w = \lambda(y)x^{k-1}w \pmod{W_{k-1}}.$$

So $[x, y]x^{k-1}w \in W^k$. Hence, the claim follows.

This proves that with respect to the basis $w, x.w, \ldots, x^{n-1}.w$, each $y \in I$ is an upper triangular matrix with diagonal entries all equal to $\lambda(y)$. In particular, $\operatorname{Tr} y = n\lambda(y)$. Now the commutator $[x, y]$ has trace zero. Thus, $0 = \operatorname{Tr}[x, y] = n\lambda([x, y])$ implies that $\lambda([x, y]) = 0$ (since we assumed $\operatorname{char} \mathbb{F} = 0$).

Finally, since L stabilizes W and $\mathbb{F}$ is algebraically closed there exists an eigenvector $v_0 \in W$ for Z, where $L = I \oplus \mathbb{F}Z$. Thus v_0 is a common eigenvector for all of L. $\square$

Note . If L is a solvable finite dimensional Lie algebra and

$$\phi : L \to \mathfrak{gl}(V)$$

is a finite dimensional representation, then Lie's Theorem applies to the image $\phi(L) \subseteq \mathfrak{gl}(V)$. In particular, Lie's Theorem provides us with information about the structure of a representation of a solvable Lie algebra.

Corollary 4.2. *Let L be a solvable (finite dimensional) Lie algebra. Then there exists a chain of ideals of L*

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = L,$$

such that $\dim L_i = i$.

Proof. Apply Lie's Theorem to the image of the adjoint representation. A subspace of L stable under the adjoint action is an ideal in L. $\square$

Corollary 4.3. *If L is a solvable (finite dimensional) Lie algebra, then $[L, L]$ is nilpotent.*

Proof. Consider the adjoint representation $\operatorname{ad} : L \to \mathfrak{gl}(L)$. By Lie's Theorem, there exists a basis such that the image $\operatorname{ad}(L) \subseteq \mathfrak{b}_n$, where $n = \dim L$. Then $\operatorname{ad}([L, L]) = [\operatorname{ad}(L), \operatorname{ad}(L)] \subseteq \mathfrak{n}_n$. Hence, $\operatorname{ad} x$ is nilpotent for any $x \in [L, L]$. By Engel's theorem, we conclude that $[L, L]$ is nilpotent. $\square$