LIE ALGEBRAS: LECTURE 4
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CRYSTAL HOYT

1. Motivation

Let L be a (finite dimensional) Lie algebra over a field $\mathbb{F}$ (where char $\mathbb{F} = 0$ and $\mathbb{F}$ is algebraically closed). Recall, that the radical of L, Rad L, is defined to be the maximal solvable ideal of L. A Lie algebra L is called semisimple if its radical is zero. Notice that $L/\text{Rad } L$ is semisimple.

A non-trivial theorem, called Levi's Theorem, states that L is semidirect product of its radical and a semisimple subalgebra. Hence, every finite dimensional Lie algebra is isomorphic to the semi-direct product of a solvable Lie algebra and a semisimple Lie algebra. This often allows one to reduce problems about general Lie algebras to problems about solvable and semisimple Lie algebras.

Last time, we saw Lie's Theorem, which gave us information about the representations of solvable Lie algebras. In this course, we will study the representation theory of complex semisimple Lie algebras.

2. Cartan's Criterion

Recall that we now assume that a Lie algebra L is finite dimensional, and that our field $\mathbb{F}$ is algebraically closed and has characteristic zero.

Theorem 2.1 (Cartan's Criterion)**.** *Let L be a subalgebra of $\mathfrak{gl}(V)$, V finite dimensional. Then L is solvable if and only if* $\text{Tr}(xy) = 0$ *for all* $x \in [L, L]$, $y \in L$.

Proof. "$\Rightarrow$" If L is solvable, then by Lie's Theorem there is a basis in which all matrices $y \in L$ are upper triangular. Then all $x \in [L, L]$ are strictly upper triangular. Thus, xy is strictly upper triangular, and $\text{Tr}(xy) = 0$.

For the proof of "$\Leftarrow$", see Humphreys pages 17-20. $\square$

Corollary 2.2. *A Lie algebra L is solvable if and only if* $\mathrm{Tr}(\mathrm{ad}\, x \cdot \mathrm{ad}\, y) = 0$ *for all* $x \in [L, L]$, $y \in L$.

Proof. By Corollary 3.4 from Lecture 3, L is solvable if and only if $\mathrm{ad}\, L$ is solvable. Now since $\mathrm{ad}[L, L] = [\mathrm{ad}\, L, \mathrm{ad}\, L]$, we have by Cartan's Criterion that $\mathrm{ad}\, L$ is solvable if and only if $\mathrm{Tr}(\mathrm{ad}\, x \cdot \mathrm{ad}\, y) = 0$ for all $x \in [L, L]$, $y \in L$. The result follows. $\square$

3. Killing form

A bilinear form B on a vector space V is a bilinear map $B : V \times V \to \mathbb{F}$. Let B be a bilinear form on $V = \mathbb{F}^n$. Then there is a matrix $M \in \mathrm{End}(\mathbb{F}^n)$ such that $B(x, y) = x^T M y$ for all $x, y \in V$. It can be computed by letting $M_{ij} = B(x_i, x_j)$ where $\{x_1, \ldots, x_n\}$ is the chosen basis for $V = \mathbb{F}^n$.

A bilinear form B is called *symmetric* if

$$B(v, w) = B(w, v)$$

for all $v, w \in V$. A bilinear form on $\mathbb{F}^n$ is symmetric if and only if the corresponding matrix is symmetric. Define the *kernel* of a symmetric bilinear form B to be

$$\mathrm{Ker} B := \{v \in V \mid B(v, w) = 0, \text{ for all } w \in V\}.$$

A symmetric bilinear form is called *nondegenerate* if its kernel is equal to zero. A symmetric bilinear form on $\mathbb{F}^n$ is nondegenerate if and only if the corresponding matrix has non-zero determinant.

Let B be a symmetric nondegenerate bilinear form on a vector space V, and let $\{v_1, \ldots, v_n\}$ be a basis for V. Then the *dual basis* of V relative to B is defined to be the basis $\{w_1, \ldots, w_n\}$ which satisfies $B(v_i, w_j) = \delta_{ij}$. In particular, $[v_1 \cdots v_n]^t M [w_1 \cdots w_n] = I$ where M is the matrix corresponding to B and I is the identity matrix. Note that the dual basis exists because M and $[v_1 \cdots v_n]^t$ are invertible, and moreover it is unique.

Let $\phi : L \to \mathfrak{gl}(V)$ be a representation of L. A bilinear form $B : V \times V \to \mathbb{F}$ is called *L-invariant* (w.r.t. the representation $\phi : L \to \mathfrak{gl}(V)$) if

$$B(\phi(x)v, w) + B(v, \phi(x)w) = 0, \text{ for all } x \in L,\ v, w \in V.$$

If $ad : L \to \mathfrak{gl}(L)$ is the adjoint representation, then this condition can be rewritten for a bilinear form $B : L \times L \to \mathbb{F}$ as

$$B([xy], z) = B(x, [yz]), \text{ for all } x, y, z \in L.$$

A bilinear form B on L satisfying this condition is called *invariant.* If B is a symmetric invariant bilinear form on L, we define the orthogonal complement of an ideal I relative to B to be

$$I^{\perp} := \{x \in L \mid B(x, y) = 0, \text{ for all } y \in I\}.$$

One can show that $I^{\perp}$ an ideal in L. Indeed, the fact that $I^{\perp}$ is a linear subspace follows from the linearity of B. Now suppose $x \in L$ and $y \in I^{\perp}$. Then for all $z \in I$, we have $[xz] \in I$ and

$$B([xy], z) = -B([yx], z) = -B(y, [xz]) = 0.$$

Hence, $[xy] \in I^{\perp}$ and so $I^{\perp}$ is an ideal. In particular, the kernel of B is an ideal of L since it is by definition equal to $L^{\perp}$.

Let L be a Lie algebra. The form

$$\kappa(x, y) := \mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$$

is called the *Killing form.* The Killing form is symmetric invariant bilinear form. The fact that κ is invariant follows from the following calculation. If $A, B, C \in \mathfrak{gl}_n$, then

$$\begin{aligned}\mathrm{Tr}([A, B]C) =& \mathrm{Tr}(ABC) - \mathrm{Tr}(B(AC)) \\ &= \mathrm{Tr}(ABC) - \mathrm{Tr}((AC)B) \\ &= \mathrm{Tr}(A[BC]).\end{aligned}$$

Hence, $\kappa([x, y], z) = \kappa(x, [y, z])$ for all $x, y, z \in L$.

Lemma 3.1. *Let I be an ideal of L. If $\kappa : L \times L \to \mathbb{F}$ is the Killing form of L and $\kappa_I : I \times I \to \mathbb{F}$ is the Killing form of I, then $\kappa_I = \kappa\mid_{I\times I}$.*

Proof. First we need a fact from linear algebra. Suppose that W is a subspace of the (finite dimensional) vector space V, and ϕ is an endomorphism of V which maps V into W, then $\mathrm{Tr}(\phi) = \mathrm{Tr}(\phi\mid_W)$. (To see this, extend a basis of W to a basis for V and consider the corresponding matrix for ϕ.) Now if $x, y \in I$, then $(\mathrm{ad}\ x\ \mathrm{ad}\ y)$ is an endomorphism of L which maps L into I, since I is an ideal. Thus, for $x, y \in I$,

$$\mathrm{Tr}((\mathrm{ad}\ x\ \mathrm{ad}\ y)) = \mathrm{Tr}((\mathrm{ad}\ x\ \mathrm{ad}\ y)\mid_I) = \mathrm{Tr}(\mathrm{ad}_I x\ \mathrm{ad}_I y).$$

Therefore, $\kappa\mid_{I\times I} = \kappa_I$. □

Theorem 3.2 (Criterion for semisimplicity)**.** *Let L be a Lie algebra. Then L is semisimple if and only if its Killing form is nondegenerate.*

Proof. Let S denote the kernel of κ.

"$\Rightarrow$" Suppose that $\mathrm{Rad}\ L = 0$. By definition, $\mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y) = 0$ for all $x \in S$ and $y \in L$, and in particular for all $y \in [S, S]$. By the corollary to Cartan's Criterion, S is solvable. Now since S is also an ideal in L, $S \subseteq \mathrm{Rad}\ L = 0$.

"$\Leftarrow$" Suppose $S = 0$. To prove that L is semisimple, it suffices to show that every abelian ideal I of L is contained in S. Suppose $x \in I$ and $y \in L$. Then $(\mathrm{ad}\ x\ \mathrm{ad}\ y)$ maps L into I and maps I to 0 (since $[I, I] = 0$). So $(\mathrm{ad}\ x\ \mathrm{ad}\ y)^2$ maps L to 0. Hence $(\mathrm{ad}\ x\ \mathrm{ad}\ y)$ is nilpotent, which implies $0 = \mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y) = \kappa(x, y)$. Hence, $I \subseteq S = 0$. Thus $\mathrm{Rad}\ L = 0$. $\square$

A Lie algebra L is said to be a *direct sum of ideals* $I_1, \ldots, I_t$ provided each I_j is an ideal in L and $L = I_1 + \cdots + I_t$ is a direct sum as subspaces. Then $[I_i, I_j] \subset I_i \cap I_j = 0$ when $i \neq j$. We write $L = I_1 \oplus \cdots \oplus I_t$, and can view L as a direct product (external direct sum) of the Lie algebras I_j.

Lemma 3.3. *Suppose L is a direct sum of ideals $L_1, \ldots, L_t$ which are simple (as Lie algebras). Then every simple ideal of L coincides with one of the L_i, and $L = [LL]$. The Lie algebra L is semisimple.*

Proof. Let I be a simple ideal of L, then $I = [II] \subset [IL] \subset I$. Then

$$I = [IL] = [IL_1] \oplus \cdots \oplus [IL_t],$$

and all but one summand must be zero since I is simple. Say $I = [IL_i]$. Then $I = [IL_i] \subset L_i$ because L_i is an ideal. Hence, $I = L_i$ since L_i is simple.

Next we show that $L = [LL]$. Now $[L_iL_j] \subset L_i \cap L_j = 0$ when $i \neq j$, and $[L_iL_i] = L_i$ since L_i is simple, thus

$$[L, L] = [L_1, L_1] \oplus \cdots \oplus [L_t, L_t] = L_1 \oplus \cdots \oplus L_t = L.$$

Now we show that each ideal J of L is a sum of certain simple ideals, $J_i = J \cap L_i \subset L_i$ so that each J_i equals L_i or 0. Clearly, $J \supseteq J_1 \oplus \cdots \oplus J_t$. Let $x \in J$ and write $x = x_1 + \cdots + x_n$ such that $x_i \in L_i$. Then if $x_i \neq 0$, we have $J_i \supset [x, L_i] = [x_i, L_i] \neq 0$ since $Z(L_i) = 0$ (L_i is simple). Then

J_i is a non-zero ideal in L_i, and hence $J_i = L_i$. Thus, $x_i \in J_i$. Therefore, $x \in J_1 \oplus \cdots \oplus J_t$. We conclude that $J = J_1 \oplus \cdots \oplus J_t$.

Finally, by applying the first part of the lemma to the ideal Rad L, we have $[\text{Rad } L, \text{Rad } L] = \text{Rad } L$. But this implies that Rad $L = 0$, since the radical of L is solvable. $\square$

Theorem 3.4. *Let L be a semisimple Lie algebra. Then there exist simple ideals $L_1, \ldots, L_t$ of L, such that $L = L_1 \oplus \ldots \oplus L_t$.*

Proof. Let I be an arbitrary ideal of L. Then $I^\perp := \{x \in L \mid \kappa(x, y) = 0, \text{ for all } y \in I\}$ is also an ideal. By the corollary of Cartan's Criterion, the ideal $I \cap I^\perp$ is solvable (since $\kappa(I \cap I^\perp, I \cap I^\perp) = 0$) and hence zero. Since the Killing form is non-degenerate we have that $\dim I + \dim I^\perp = \dim L$, and we conclude that $L = I \oplus I^\perp$.

The proof is by induction on the dimension of L. If L is simple, then we are done. Otherwise, take a minimal nonzero ideal L_1. Then $L = L_1 \oplus L_1^\perp$. Suppose I is an ideal in L_1. Let $x \in L = L_1 \oplus L_1^\perp$ and $y \in I$. Write $x = a + b$ where $a \in L_1$ and $b \in L_1^\perp$. Then $[x, y] = [a + b, y] = [a, y] + [b, y] = [a, y] \in I$ since I is an ideal in L_1. Hence, I is an ideal in L. Thus, L_1 is simple by minimality. Also, $L_1^\perp$ is semisimple by the same argument, since a non-trivial solvable ideal in $L_1^\perp$ would be a non-trivial solvable ideal in L. Thus, we can apply the induction hypothesis to $L_1^\perp$ to receive the desired decomposition. $\square$

Corollary 3.5. *If L is semisimple then $L = [LL]$, and all ideals and homomorphic images of L are semisimple. Moreover, each ideal is a sum of certain simple ideals of L.*

Remark 3.6. We have shown that a Lie algebra is semisimple if and only if it is a direct product of simple Lie algebras.

4. Automorphisms

Let L be a Lie algebra. Then an *automorphism* of L is a Lie algebra homomorphism which is an isomorphism of L onto itself.

Example 4.1. Suppose L is a subalgebra of $\mathfrak{gl}(V)$. Let $GL(V)$ to be the group of invertible endomorphisms of a vector space V. If $g \in GL(V)$ and if $gLg^{-1} = L$, then the map $x \mapsto gxg^{-1}$ is an automorphism of L. For example, if $L = \mathfrak{gl}(V)$ or $L = \mathfrak{sl}(V)$ then the condition $gLg^{-1} = L$ holds for any $g \in GL(V)$.