LIE ALGEBRAS: LECTURE 5
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1. Modules

Let L be a Lie algebra. A vector space V with an operation

$$L \times V \to V$$
$$(x, v) \mapsto x.v$$

is called an *L-module* if for all $x, y \in L$, $v, w \in V$, $a, b \in \mathbb{F}$ the following conditions hold:

(1) (ax + by).v = a(x.v) + b(y.v)
(2) x.(av + bw) = a(x.v) + b(x.w)
(3) [xy].v = x.y.v - y.x.v.

Example 1.1. If $\phi : L \to \mathfrak{gl}(V)$ is a representation of L, then V is an L-module via the action $x.v = \phi(x)v$. Conversely, if V is an L-module then $\phi(x)v = x.v$ defines a representation.

Fix a Lie algebra L, and let V and W be L-modules. An *L-module homomorphism* is a linear map $f : V \to W$ such that $f(x.v) = x.f(v)$ for all $x \in L$ and $v \in V$. The kernel of this map is an L-submodule of V.

If $f : V \to W$ is an L-module homomorphism and is an isomorphism of vector spaces, then $f^{-1} : W \to V$ is also an L-module homomorphism. In this case, we call f an *isomorphism* of L-modules, and the modules V and W are called *equivalent representations* of L.

An L-module V is called *irreducible* (or *simple*) if it has precisely two L-submodules: itself and 0. A *direct sum* of L-modules $V_1, \ldots, V_t$ is a direct sum of vector spaces $V_1 \oplus \cdots \oplus V_t$, with the action of L defined:

$$x.(v_1, v_2, \ldots, v_t) = (x.v_1, x.v_2, \ldots, x.v_t).$$

An L-module V is called *completely reducible* (or *semisimple*) if V is a direct sum of irreducible of L-submodules. If V is finite dimensional, then this is equivalent to the condition that each L-submodule W of V has a compliment submodule W' such that $V = W \oplus W'$.

Example 1.2. If L is a one dimensional Lie algebra, say $L = \mathbb{F}x$. Then the module V given by the representation $\phi : L \to \mathfrak{gl}_2$ with

$$\phi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not completely reducible. Indeed, let $\{v_1, v_2\}$ be a basis for the vector space V. Then v_1 is an eigenvector for $\phi(x)$ with eigenvalue 0. So $\mathbb{F}v_1$ is an L-submodule of V. Now suppose $M = \mathbb{F}(av_1 + bv_2)$ is the 1-dimensional compliment submodule to $\mathbb{F}v_1$, so that $V = \mathbb{F}v_1 \oplus M$ is a direct sum of modules. Then $b \neq 0$, since $\{v_1, (av_1 + bv_2)\}$ is a vector space basis for V. But $\phi(x)(av_1 + bv_2) = bv_1$, so M is not a submodule. Hence, the module V is not completely reducible.

Example 1.3. Two representations $\phi_1 : L \to \mathfrak{gl}(V)$ and $\phi_2 : L \to \mathfrak{gl}(W)$ are equivalent if there exists a linear isomorphism $f : V \to W$ such that $f(x.v) = x.f(v)$ for all $v \in V$, or equivalently, such that $f(\phi_1(x)v) = \phi_2(x)f(v)$ for all $x \in L$, $v \in V$. Thus, the representations ϕ_1 and ϕ_2 are equivalent if and only if there exists a linear isomorphism $f : V \to W$ such that $f\phi_1(x)f^{-1} = \phi_2(x)$ for all $x \in L$. So two representations are equivalent when one is obtained from the other by conjugation, i.e. a change a basis if $V = W$ as vector spaces.

Let L be a one dimensional Lie algebra, say $L = \mathbb{F}x$. Then a representation $\phi : L \to \mathfrak{gl}(V)$ is determined by the image of x, i.e. it is determined by the endomorphism $\phi(x) \in \mathfrak{gl}(V)$. Hence, the classification of n-dimensional representations is equivalent to the classification of square matrices up to conjugation. When the base field $\mathbb{F}$ is algebraically closed, there is a bijection between isomorphism classes of n-dimensional representations and $n \times n$-matrices in Jordan normal form. In this case, simple modules are one dimensional and finite dimensional modules are not completely reducible. Why? Because, an upper triangular matrix preserves a flag of submodules, so a module can only be simple if it is one dimensional. Also, not all finite dimensional modules are completely reducible, since not all matrices are diagonalizable.

Example 1.4. Let L be a solvable Lie algebra over a field $\mathbb{F}$ which is algebraically closed and has characteristic zero. Then all irreducible finite dimensional representations of L are 1-dimensional. This follows immediately by applying Lie's Theorem to the image of the representation, $\phi(L) \subset \mathfrak{gl}_n$. If $n \neq 1$, then we have a non-trivial flag of submodules.

Denote by $\mathrm{Hom}_L(V, W)$ the collection of all L-module homomorphisms from V to W. This is a vector space over $\mathbb{F}$, since if $a \in \mathbb{F}$, $f, g \in \mathrm{Hom}_L(V, W)$ then $af,\ f + g \in \mathrm{Hom}_L(V, W)$, where $(f + g)(v) := f(v) + g(v)$.

Lemma 1.5 (Schur's Lemma). *Suppose that the base field $\mathbb{F}$ is algebraically closed. Let V be a simple finite dimensional module over a Lie algebra L. Then $Hom_L(V, V) = \mathbb{F} \cdot id$. Thus, the only endomorphisms of V commuting with $\phi(L)$ are the scalars.*

Proof. Let $f \in \mathrm{Hom}_L(V, V)$. For any $c \in \mathbb{F}$ the linear operator $(f - c \cdot id)$ is an L-module homomorphism. It must be either injective or zero, since the kernel is a submodule of V. If $f : V \to V$ is injective then it is an isomorphism, since V is finite dimensional. Thus, $(f - c \cdot id)$ is either an isomorphism or zero. If c is an eigenvalue of f, then $(f - c \cdot id)$ has a non-zero kernel, implying $(f - c \cdot id) = 0$. Hence, $f = c \cdot id$ for some $c \in \mathbb{F}$, since $\mathbb{F}$ is algebraically closed. If $f : V \to V$ is an endomorphism of the simple module V such that $f \circ \phi(x) = \phi(x) \circ f$ for all $x \in L$, then $f = c \cdot id$ for some $c \in \mathbb{F}$. □

Given an L-module V we construct the *dual module* using the dual vector space V^*. We define an action of L on V^* as follows: for $f \in V^*$, $x \in L$, $v \in V$, let $(x.f)(v) = -f(x.v)$. One can check that $([xy].f)(v) = ((x.y - y.x).f)(v)$.

2. Casimir element

Let L be a semisimple Lie algebra, and let $\phi : L \to \mathfrak{gl}(V)$ be a representation. Define a symmetric invariant bilinear form $\beta(x, y) = \mathrm{Tr}(\phi(x)\phi(y))$ on L, called the *trace form*. The kernel of $\beta(x, y)$, denote Ker β, is a ideal in L. If ϕ is a faithful representation, then $\beta(x, y)$ is nondegenerate. Indeed, by Cartan's Criterion, $\phi(\text{Ker } \beta)$ is a solvable ideal in $\phi(L)$. If ϕ is a faithful representation, so that $L \cong \phi(L) \subset \mathfrak{gl}(V)$, then Ker β ($\cong \phi(\text{Ker } \beta)$) is a solvable ideal in L ($\cong \phi(L)$). Since L is semisimple this implies Ker $\beta = 0$.

Let L be a Lie algebra, and let β be a nondegenerate symmetric invariant bilinear form on L. Let $\{x_1, \ldots, x_n\}$ be a basis of L, and let $\{y_1, \ldots, y_n\}$ be dual basis relative to β, i.e. $\beta(x_i, y_j) = \delta_{ij}$. The dual basis exists when $\beta(x, y)$ is symmetric nondegenerate, because the corresponding matrix M_B is invertible, so we can solve the following system for the vectors y_i:

$$\begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix} M_B \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix} = I.$$

For each $x \in L$, we can write $[x, x_i] = \sum_j a_{ij} x_j$ and $[x, y_i] = \sum_j b_{ij} y_j$. Then we can show that $a_{ik} = -b_{ki}$ using the fact that β is invariant.

$$\begin{aligned} a_{ik} &= \sum_j a_{ij}\delta_{jk} = \sum_j a_{ij}\beta(x_j, y_k) \\ &= \beta([x, x_i], y_k) = -\beta([x_i, x], y_k) = -\beta(x_i, [x, y_k]) \\ &= -\sum_j b_{kj}\beta(x_i, y_j) = -b_{ki} \end{aligned}$$

Let $\phi : L \to \mathfrak{gl}(V)$ be a representation of L, and let

$$c_\phi(\beta) = \sum_i \phi(x_i)\phi(y_i) \in \text{End }(V).$$

Now if $x, y, z \in \mathfrak{gl}(V)$, then $[x, yz] = [x, y]z + y[x, z]$. So

$$\begin{aligned}[\phi(x), c_\phi(\beta)] &= \sum_i [\phi(x), \phi(x_i)]\phi(y_i) \ + \ \sum_i \phi(x_i)[\phi(x), \phi(y_i)] \\ &= \sum_{i,j} a_{ij}\phi(x_j)\phi(y_i) \ + \ \sum_{i,j} b_{ij}\phi(x_i)\phi(y_j) \\ &= 0.\end{aligned}$$

Thus, $c_\phi(\beta)$ is an endomorphism of V which commutes with $\phi(L)$.

Now suppose $\phi : L \to \mathfrak{gl}(V)$ is a faithful representation, so that the trace form $\beta(x, y) = \mathrm{Tr}(\phi(x), \phi(y))$ is nondegenerate. Fix a basis $\{x_1, \ldots, x_n\}$. Then c_ϕ is called the *Casimir element* of ϕ, and

$$\mathrm{Tr}(c_\phi) = \sum_i \mathrm{Tr}(\phi(x_i)\phi(y_i)) = \sum_i \beta(x_i, y_i) = \dim L.$$

If ϕ is an irreducible representation, then by Schur's Lemma, c_ϕ is a scalar (equal to $\dim L / \dim V$), and so is independent of the choice of basis.

Example 2.1. Let $L = sl_2(\mathbb{F})$ and consider the natural representation $\phi : L \to \mathfrak{gl}(V)$, $\dim V = 2$. The dual basis with respect to the trace form of the standard basis $\{e, h, f\}$ is: $\{f, h/2, e\}$. So $c_\phi = ef + \frac{1}{2}hh + fe = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$.

If L is semisimple but $\phi : L \to \mathfrak{gl}(V)$ is not faithful, then $L = \ker\phi \oplus L'$. Then the restriction of ϕ to L' is a faithful representation with $\phi(L) = \phi(L')$. The Casimir element c_ϕ of $\phi : L \to \mathfrak{gl}(V)$ is defined to be the Casimir element of $\phi : L' \to \mathfrak{gl}(V)$. It commutes with $\phi(L)$ since $\phi(L) = \phi(L')$.